Polynomials

Concept of Polynomials
Polynomials

Consider this situation involving trains.

The speed of an express train is ten less than twice that of a passenger train. If each travels for as many hours as its speed, then what is the difference between the distances travelled by them?

Let the speed of the passenger train be x km/hr.

Then, travelling time of the train = x hours

Distance travelled by it = Speed × Time = $x \times x = x^2$ km

Now, speed of the express train = $(2x - 10)$ km/hr

Its travelling time = $(2x - 10)$ hours

Distance travelled by it = $(2x - 10)(2x - 10) = (4x^2 - 40x + 100)$ km

Thus, required difference = $4x^2 - 40x + 100 - x^2 = (3x^2 - 40x + 100)$ km

The expression $3x^2 - 40x + 100$ is an example of a polynomial. Different real-life problems such as the one given above can be expressed in the form of polynomials. Go through this lesson to familiarize yourself with these useful expressions.

Topics to be covered in this lesson:

- Identifying polynomials
- Constant polynomials
- Classification of polynomials according to the number of terms

- **Did You Know?**
- Ancient Babylonians developed a unique system to calculate things using formulae. These formulae consisted of letters, mathematical operators (+, −, ×, ÷) and numbers. It was this system that led to the development of algebra. The word 'algebra' is derived from the Arabic word 'al-jabr' meaning 'the reunion of broken parts'. Another Arabian connection with algebra is the Arab mathematician Muhammad ibn Musa al-Khwarizmi, whose theories greatly influenced this branch of mathematics.

Solved Examples

Easy

Example 1:

Which of the following expressions are polynomials? Justify your answers.

i) $2x^{1/2} + 3x + 4$

ii) $8x^3 + 7 + x$

iii) $2x^3 - \sqrt{2}y^5$

iv) $\sqrt{121}x - \frac{2}{3}y^2 + x^3$

v) $5\left(\sqrt{x}\right)^2 + 9x^2$

vi) $\frac{14}{x^2} - 9x^2$

Solution:

i) $2x^{1/2} + 3x + 4$

This expression is not a polynomial because the exponent of the first term is $\frac{1}{2}$, which is not a whole number.

ii) $8x^3 + 7 + x$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variable x are whole numbers.

iii) $2x^3 - \sqrt{2}y^5$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variables x and y are whole numbers.

iv) $\sqrt{121}x - \frac{2}{3}y^2 + x^3 = 11x - \frac{2}{3}y^2 + x^3$

This expression is a polynomial as all the coefficients are real numbers and the exponents of the variables x and y are whole numbers.

v) $5\left(\sqrt{x}\right)^2 + 9x^2$

This expression is a polynomial because it can be written as $5x + 9x^2$, in which all the coefficients are real numbers and the exponents of the variable x are whole numbers.

vi) $\frac{14}{x^2} - 9x^2$

This expression is not a polynomial because it can be written as $14x^{-2} - 9x^2$, in which the variable in the first term has a negative exponent.

Example 2:

For each of the given polynomials, state whether it is a monomial, binomial or trinomial.

i) $4x^3$

ii) $13y^5 - y$

iii) $29t^3 + 14t - 9$

iv) $x - x^2$

Solution:

i) $4x^3$ is a monomial as it has only one term.

ii) $13y^5 - y$ is a binomial as it has two terms.

iii) $29t^3 + 14t - 9$ is a trinomial as it has three terms.

iv) $x - x^2$ is a binomial as it has two terms.

Medium

Example 1:

For each of the given polynomials, state whether it is a monomial, binomial or trinomial.

i) $(4x^3 + 3y) - (3x^3 + x^2) + (y - x^3)$

ii) $(t^2 + 1)^2$

iii) $[(m - 1)(m + 1)] + 2m^2 + 1$

Solution:

i) $(4x^3 + 3y) - (3x^3 + x^2) + (y - x^3)$

$= 4x^3 + 3y - 3x^3 - x^2 + y - x^3$

$= (4x^3 - 3x^3 - x^3) - x^2 + (3y + y)$

$= - x^2 + 4y$

The polynomial can be reduced to $-x^2 + 4y$, which has two terms; so, it is a binomial.

ii) $(t^2 + 1)^2$

$= (t^2)^2 + 2t^2 + 1^2 [\because (a + b)^2 = a^2 + 2ab + b^2]$

$= t^4 + 2t^2 + 1 [\because (a^m)^n = a^{m \times n}]$

The polynomial can be reduced to $t^4 + 2t^2 + 1$, which has three terms; so, it is a trinomial.

iii) $[(m - 1)(m + 1)] + 2m^2 + 1$

$= (m^2 - 1) + 2m^2 + 1 \; [\because (a - b)(a + b) = a^2 - b^2]$

$= m^2 + 2m^2 - 1 + 1$

$= 3m^2$

The polynomial can be reduced to $3m^2$, which has only one term; so, it is a monomial.

Hard

Example 1:

State whether or not the following expressions are polynomials. Justify your answers.

i) $\frac{x^3}{xy^{-2}} + \frac{xy}{10} - \frac{1}{7} - \frac{\sqrt{2}y^2}{y^{-1}}$

ii) $\frac{y^2}{3} + \frac{14x^{-4}}{x^2} - 5xy + \frac{x}{2}$

Solution:

i) $\frac{x^3}{xy^{-2}} + \frac{xy}{10} - \frac{1}{7} - \frac{\sqrt{2}y^2}{y^{-1}}$

$$= x^3(x^{-1}y^2) + \frac{1}{10}xy - \frac{1}{7} - \sqrt{2}y^2(y)$$
$$= x^{3-1}y^2 + \frac{1}{10}xy - \frac{1}{7} - \sqrt{2}y^{2+1}$$
$$= x^2y^2 + \frac{1}{10}xy - \frac{1}{7} - \sqrt{2}y^3$$

In the reduced form of the given expression, all the coefficients are real numbers and the exponents of the variables x and y are whole numbers. Hence, the given expression is a polynomial.

ii) $\frac{y^2}{3} + \frac{14x^{-4}}{x^2} - 5xy + \frac{x}{2}$

$$=\frac{1}{3}y^2+14x^{-4}(x^{-2})-5xy+\frac{1}{2}x$$

$$=\frac{1}{3}y^2+14x^{-4-2}-5xy+\frac{1}{2}x$$

$$=\frac{1}{3}y^2+14x^{-6}-5xy+\frac{1}{2}x$$

In the reduced form of the given expression, the exponent of the second term (i.e., −6) is not a whole number. Hence, the given expression is not a polynomial.

- **Did You Know?**

The word 'polynomial' is a combination of the Greek words 'poly' meaning 'many' and 'nomos' meaning 'part or portion'. Thus, a polynomial is an algebraic expression having many parts.

Solved Examples

Easy

Example 1:

Find the coefficient of *x* in the following polynomials.

i) $\frac{\pi}{2}x^3-3x^2$

ii) $(2x^2-x)+7+3x$

Solution:

i) $\frac{\pi}{2}x^3-3x^2$

This expression can also be written as $\frac{\pi}{2}x^3-3x^2+0.x$.

Thus, in the given polynomial, the coefficient of x is 0.

ii) $(2x^2-x)+7+3x$

$=2x^2-x+7+3x$

$= 2x^2 + 2x + 7$

Thus, in the given polynomial, the coefficient of x is 2.

Medium

Example 1:

For each of the following polynomials, write the constant term and the coefficient of the variable having the highest exponent.

i) $-\left(x^4\right)^{\frac{1}{2}} - 3\left(\sqrt{x^3}\right)^2 + (x + 2) - 4$

ii) $3(y + 2)^2 - 5y(y^2 + 2/y) + y^3$

Solution:

i) $-\left(x^4\right)^{\frac{1}{2}} - 3\left(\sqrt{x^3}\right)^2 + (x + 2) - 4$

$= -x^{4\times\frac{1}{2}} - 3x^3 + (x + 2) - 4$ [$\because (a^m)^n = a^{m \times n}$ and $\left(\sqrt{a}\right)^2 = a$]

$= -x^2 - 3x^3 + x - 2$

$= -3x^3 - x^2 + x - 2$

In this reduced form of the given polynomial, we have:

Constant term $= -2$

Term with the highest exponent $= -3x^3$

So, coefficient of the variable having the highest exponent $= -3$

ii) $3(y + 2)^2 - 5y\left(y^2 + \frac{2}{y}\right) + y^3$

$= 3(y^2 + 4y + 4) - 5y^3 - 10 + y^3$

$= 3y^2 + 12y + 12 - 4y^3 - 10$

$= -4y^3 + 3y^2 + 12y + 2$

In this reduced form of the given polynomial, we have:

Constant term = 2

Term with the highest exponent = $-4y^3$

So, coefficient of the variable having the highest exponent = −4

Hard

Example 1:

Find the coefficients of x^3, x^2, x, y^3, y^2 and y in the following polynomials. Also, find the real constants.

i) $17x^4 - 3y - x(2x^3 + x) + 3(2y - 4x^2 + 1) - 4$

ii) $y^4\left(\frac{3}{2y} - 2\right) - \frac{1}{2}(6x^2 + y + 1) + y - 4x^2$

Solution:

i) $17x^4 - 3y - x(2x^3 + x) + 3(2y - 4x^2 + 1) - 4$

$= 17x^4 - 3y - 2x^4 - x^2 + 6y - 12x^2 + 3 - 4$

$= 15x^4 - 13x^2 + 3y - 1$

This reduced form of the given polynomial can be further written as:

$15x^4 + 0.x^3 - 13x^2 + 0.x + 0.y^3 + 0.y^2 + 3y - 1$

Therefore, we have:

Coefficient of x^3 = 0 Coefficient of x^2 = −13 Coefficient of x = 0

Coefficient of y^3 = 0 Coefficient of y^2 = 0 Coefficient of y = 3

Real constant = −1

ii) $y^4\left(\frac{3}{2y}-2\right)-\frac{1}{2}(6x^2+y+1)+y-4x^2$

$=\frac{3}{2y}\times y^4-2y^4-\frac{1}{2}\times 6x^2-\frac{1}{2}\times y-\frac{1}{2}+y-4x^2$

$=\frac{3}{2}y^3-2y^4-3x^2-\frac{1}{2}y+y-4x^2-\frac{1}{2}$

$=\frac{3}{2}y^3-2y^4-7x^2+\frac{1}{2}y-\frac{1}{2}$

$=-7x^2-2y^4+\frac{3}{2}y^3+\frac{1}{2}y-\frac{1}{2}$

This reduced form of the given polynomial can be further written as:

$0.x^3-7x^2+0.x-2y^4+\frac{3}{2}y^3+0.y^2+\frac{1}{2}y-\frac{1}{2}$

Therefore, we have:

Coefficient of x^3 = 0 Coefficient of x^2 = −7 Coefficient of x = 0

Coefficient of y^3 = $\frac{3}{2}$ Coefficient of y^2 = 0 Coefficient of y = $\frac{1}{2}$

Real constant = $-\frac{1}{2}$

Different Forms of a Polynomial

- A polynomial can found and written in different forms. These forms are explained below.
- **Standard form:** If the terms of a polynomial are written in descending order or ascending order of the powers of the variables then the polynomial is said to be in the standard form.
- For example, the polynomial $3x + 15x^4 - 1 - 13x^2$ is not in the standard form. It can be written in the standard form as $15x^4 - 13x^2 + 3x - 1$ or $-1 + 3x - 13x^2 + 15x^4$.
- **Index form:** Observe the polynomial $x^6 - 2x^4 - 10x^3 + 5$. In this polynomial, terms having x^5, x^2 and x are missing. These terms can be added to the polynomial with

coefficient 0. Thus, the obtained polynomial will be $x^6 + 0x^5 - 2x^4 - 10x^3 + 0x^2 + 0x + 5$.

- The polynomial obtained on adding the missing terms is said to be in the index form.

- **Coefficient form:** When the coefficients of all the terms of a polynomial are written in a bracket by separating with comma then the polynomial is said to be written in the coefficient form.

- It should be noted that if a term is missing then its coefficient is taken as 0. So, it is better to write the given polynomial in the index form before wrting it in the coefficient form.

For example, to write the polynomial $x^6 - 2x^4 - 10x^3 + 5$ in the coefficient form, we will first write it in the index form as $x^6 + 0x^5 - 2x^4 - 10x^3 + 0x^2 + 0x + 5$.

Now, it can be written in the coefficient form as $(1, 0, -2, -10, 0, 0, 5)$.

- **Solved Examples**

- **Example 1:** Express the given polynomials in the standard form.

(i) $-2y^3 + 5y^5 - 2 + y$

(ii) $11a - a^6 - 2a^3 + a^2 - 1$

Solution:

We know that if the terms of a polynomial are written in descending order or ascending order of the powers of the variables then the polynomial is said to be in the standard form.

- Given polynomials can be written in the standard form as follows:

(i) **Given form:** $-2y^3 + 5y^5 - 2 + y$

- **Standard form:** $5y^5 - 2y^3 + y - 2$ or $-2 + y - 2y^3 + 5y^5$

(ii) **Given form:** $11a - a^6 - 2a^3 + a^2 - 1$

- **Standard form:** $-a^6 - 2a^3 + a^2 + 11a - 1$ or $-1 + 11a + a^2 - 2a^3 - a^6$

Example 2: Express the given polynomials in the index form and coefficient form.

- (i) $2m^7 + 12m^5 - 7m^2 - m$
- (ii) $-4p^6 + 3p^3 - 2p + 7$

Solution:

- We know that the polynomial obtained on adding the missing terms is said to be in the index form.
- Also, when the coefficients of all the terms of a polynomial are written in a bracket by separating with comma then the polynomial is said to be written in the coefficient form.
- Given polynomials can be written in the index form and coefficient form as follows:

(i) **Given form:** $2m^7 + 12m^5 - 7m^2 - m$

- **Index form:** $2m^7 + 0m^6 + 12m^5 + 0m^4 + 0m^3 - 7m^2 - m$
- **Coefficient form:** (2, 0, 12, 0, 0, −7, −1)

(ii) **Given form:** $-4p^6 + 3p^3 - 2p + 7$

- **Index form:** $-4p^6 + 0p^5 + 0p^4 + 3p^3 + 0p^2 - 2p + 7$
- **Coefficient form**: (−4, 0, 0, 3, 0, −2, 7)
- **Example 3:** Express the given coefficient forms in the index forms by taking *x* as the variable.
- (i) (10, 0, −2, 1, 0, 0, 7)
- (ii) (−1, 2, 0, 3, 0, 6)

Solution:

Given polynomials can be written in the index form as follows:

(i) **Given form:** (10, 0, −2, 1, 0, 0, 7)

Index form: $10x^6 + 0x^5 - 2x^4 + x^3 + 0x^2 + 0x + 7$

(ii) **Given form:** (−1, 2, 0, 3, 0, 6)

Index form: $-x^5 + 2x^4 + 0x^3 + 3x^2 + 0x + 6$

Degree of Polynomial

More about Polynomials

We know that a polynomial comprises a number of terms, which may have variables or numbers or both. Also, each term can be represented with a variable having some **exponent**

. Exponents of the variables in a given polynomial can be the same or different.

Let us consider a polynomial $2x^5 + 4x^2 + 9$.

The terms of this polynomial and their exponents are as follows:

First term = $2x^5$; exponent in the first term = 5

Second term = $4x^2$; exponent in the second term = 2

Third term = 9 = $9x^0$; exponent in the third term = 0

Note that all the exponents in the above polynomial are different. These exponents help us to identify the degrees of polynomials. Polynomials are categorized based on their degrees.

In this lesson, we will learn about the degrees of polynomials and the classification of polynomials based on the same.

Whiz Kid

When a polynomial has an equals sign (=), then it becomes an equation. The maximum number of solutions of an equation is less than or equal to the degree of that equation.

Solved Examples

Example 1: Find the degree of each term of the polynomial $3x^6 + 3x^4 - 6x + 3$. Also find the degree of the polynomial.

Solution:

The degree of the term $3x^6$ is 6.

The degree of the term $3x^4$ is 4.

The degree of the term $-6x$ is 1.

The degree of the term 3 is 0.

Here, the highest degree is 6. Hence, the degree of the polynomial is 6.

Medium

Example 1: **Write the degree of each of the following polynomials.**

i) $\frac{x^2}{2x} - 9x^7 + \frac{1}{x^{-4}} + 7$

ii) $\frac{x^2}{3} + \frac{4x^{-1}}{x^{-2}} - 5x^2 + \frac{x}{2} - 9$

Solution:

i) $\frac{x^2}{2x} - 9x^7 + \frac{1}{x^{-4}} + 7$

$= \frac{x^{2-1}}{2} - 9x^7 + x^4 + 7 \qquad \left(\because \frac{a^m}{a^n} = a^{m-n}, \text{where } m > n;\ \frac{1}{a^{-m}} = a^m\right)$

$= \frac{x}{2} - 9x^7 + x^4 + 7$

In the given polynomial, the highest degree is 7. Hence, the degree of the polynomial is 7.

ii) $\frac{x^2}{3} + \frac{4x^{-1}}{x^{-2}} - 5x^2 + \frac{x}{2} - 9$

$= \frac{x^2}{3} + 4x^{2-1} - 5x^2 + \frac{x}{2} - 9 \qquad \left(\because \frac{a^m}{a^n} = a^{n-m}, \text{where } m < n\right)$

$= \frac{x^2}{3} + 4x - 5x^2 + \frac{x}{2} - 9$

$= -\frac{14x^2}{3} + \frac{9x}{2} - 9$

In the given polynomial, the highest degree is 2. Hence, the degree of the polynomial is 2.

The Degree of a Polynomial in more than one Variable

In case of the polynomials in one variable, the degree of a polynomial is the highest exponent of the variable in the polynomial, but what about the degree of the polynomial in more than one variable?

In this case, the sum of the powers of all variables in each term is obtained and the highest sum among all is the degree of the polynomial.

For example, find the degree of the polynomial $2xy + 3y^2z + 4x^2yz^2 - xyz - 2x^3$. Let us find the sum of the powers of all variables in each term of this polynomial.

Sum of the powers of all variables in the term $2xy = 1 + 1 = 2$

Sum of the powers of all variables in the term $3y^2z = 2 + 1 = 3$

Sum of the powers of all variables in the term $4x^2yz^2 = 2 + 1 + 2 = 5$

Sum of the powers of all variables in the term $-xyz = 1 + 1 + 1 = 3$

Sum of the powers of all variables in the term $-2x^3 = 3$

Among all the sums, 5 is the highest and thus, the degree of the polynomial $2xy + 3y^2z + 4x^2yz^2 - xyz - 2x^3$ is 5.

Similarly, we can find the degree of any polynomial in more than one variable.

Solved Examples

Example 1: **Write the degree of each of the following polynomials.**

(i) $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$

(ii) $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$

Solution:

(i) Let us find the sum of the powers of all variables in each term of $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$.

Sum of the powers of all variables in the term $24a^2b = 2 + 1 = 3$

Sum of the powers of all variables in the term $-abc = 1 + 1 + 1 = 3$

Sum of the powers of all variables in the term $11abc^2 = 1 + 1 + 2 = 4$

Sum of the powers of all variables in the term ab^2c^3= 1 + 2 + 3 = 6

Sum of the powers of all variables in the term $-7a^2b^2$ = 2 + 2 = 4

Among all the sums, 6 is the highest and thus, the degree of the polynomial $24a^2b - abc + 11abc^2 + ab^2c^3 - 7a^2b^2$ is 6.

(ii) Let us find the sum of the powers of all variables in each term of $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$.

Sum of the powers of all variables in the term $5p^5$ = 5

Sum of the powers of all variables in the term $10p^2qr$ = 2 + 1 + 1 = 4

Sum of the powers of all variables in the term $-9p^2qr^2$ = 2 + 1 + 2 = 5

Sum of the powers of all variables in the term $-p^2r^2$ = 2 + 2 = 4

Sum of the powers of all variables in the term $2pq^2$ = 1 + 2 = 3

Sum of the powers of all variables in the term $-2p^3q^2r$ = 3 + 2 + 1 = 6

Among all the sums, 6 is the highest and thus, the degree of the polynomial $5p^5 + 10p^2qr - 9p^2qr^2 - p^2r^2 + 2pq^2 - 2p^3q^2r$ is 6.

Whiz Kid

If all the terms in a polynomial have the same exponent, then the expression is referred to as a **homogenous polynomial**.

Did You Know?

The graphs oflinear polynomials are always straight lines. This is why these polynomials are called ‘linear’ polynomials.

Solved Examples

Example 1: Classify each of the given polynomials according to its degree.

i) $11x^3 + 7x + 3$

ii) $8x^2 + 3x$

iii) $x + 5$

iv) $9t^3$

Solution:

i) $11x^3 + 7x + 3$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

ii) $8x^2 + 3x$

The degree of this polynomial is 2. Hence, it is a quadratic polynomial.

iii) $x + 5$

The degree of this polynomial is 1. Hence, it is a linear polynomial.

iv) $9t^3$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

Example 2: Give an example of each of the following polynomials.

i) A monomial of degree 50

ii) A binomial of degree 17

iii) A trinomial of degree 99

Solution:

i) A monomial of degree 50 means a polynomial having one term and 50 as the highest exponent. An example of such a polynomial is $23y^{50}$.

ii) A binomial of degree 17 means a polynomial having two terms and 17 as the highest exponent. An example of such a polynomial is $41t^{17} + 53t$.

iii) A trinomial of degree 99 means a polynomial having three terms and 99 as the highest exponent. An example of such a polynomial is $p^{99} + 5p - 12$.

Medium

Example 1: Classify each of the given polynomials according to its degree.

i) $\frac{x^2}{3}+4x^3-(5x^2+4x^3)+\frac{x}{2}-9$

ii) $x + 3x^2 + (x + 2)(x^2 + 4 - 2x) + 54$

Solution:

i) $\frac{x^2}{3}+4x^3-(5x^2+4x^3)+\frac{x}{2}-9$

$=\frac{x^2}{3}+4x^3-5x^2-4x^3+\frac{x}{2}-9$

$=\frac{x^2}{3}-5x^2+\frac{x}{2}-9$

$=-\frac{14x^2}{3}+\frac{x}{2}-9$

The degree of this polynomial is 2. Hence, it is a quadratic polynomial.

ii) $x + 3x^2 + (x + 2)(x^2 + 4 - 2x) + 54$

$= x + 3x^2 + (x^3 + 2^3) + 54 \quad [\because a^3 + b^3 = (a + b)(a^2 + b^2 - ab)]$

$= x + 3x^2 + x^3 + 8 + 54$

$= x + 3x^2 + x^3 + 62$

The degree of this polynomial is 3. Hence, it is a cubic polynomial.

Values of Polynomials at Different Points

Value of a Polynomial at Different Points

What do you observe when a ball is dropped from a height? It bounces again and again until it comes to rest after some time. Also, the height to which the ball bounces keeps decreasing and then becomes zero.

Suppose a ball dropped from a certain height h bounces to two-third of that height. If this height of bounce is H, then we can say that $H = \frac{2}{3}h$. In this equality, the value of H changes when the value of h undergoes a change, i.e., the value of H depends upon the value of h.

Polynomials can be described in a similar manner. When the value of the variable in a polynomial changes, the value of the polynomial also undergoes a change. This means that the value of a polynomial depends upon the value of the variable present in it.

In this lesson, we will learn to find the value of a given polynomial at different points.

Finding the Value of a Polynomial at Different Points

Consider the polynomial, $p(x) = x^2 - 4x + 5$. The variable here is x.

Hence, for the different values of the variable x, we get different values of the polynomial $p(x)$.

Let us find the value of this polynomial at $x = 2$. We will do so by replacing x with 2 in the given polynomial.

$p(2) = 2^2 - 4 \times 2 + 5$

$\Rightarrow p(2) = 4 - 8 + 5$

$\Rightarrow \therefore p(2) = 1$

So, the value of the polynomial is 1 when $x = 2$.

Now, let us see what happens on replacing x by −3 in the given polynomial.

$p(-3) = (-3)^2 - 4\,(-3) + 5$

$\Rightarrow p(-3) = 9 + 12 + 5$

$\Rightarrow \therefore p(-3) = 26$

The value of the polynomial is different this time. We can see that the value of the polynomial is 26 when $x = -3$. This shows that the value of the polynomial changes with the change in the value of the variable in it.

Similarly, we can find the value of any polynomial for any value of the variable involved.

Did You Know?

- **The sum of the coefficients of a polynomial $p(x)$ is equal to $p(1)$.**

  For example, let us consider a polynomial, $p(x) = x^2 - 4x + 5$.

Here, sum of the coefficients of $p(x) = p(1) = 1^2 - 4(1) + 5 = 2$

- **The constant coefficient of a polynomial $p(x)$ is equal to $p(0)$.**

  Let us consider the same polynomial as above.

  Here, constant coefficient of $p(x) = p(0) = 0^2 - 4(0) + 5 = 5$

Whiz Kid

The plotting of the consecutive values of the variable and the polynomial on the coordinate plane gives a curve. Thus, each polynomial represents a curve.

Solved Examples

Easy

Example 1: Find the value of the polynomial $x^2 + 5x + 9$ at $x = -2$.

Solution:

Let $p(x) = x^2 + 5x + 9$

On replacing x with −2, we get:

$p(-2) = (-2)^2 + 5(-2) + 9$

$= 4 - 10 + 9$

$= 3$

Thus, at $x = -2$, the value of $x^2 + 5x + 9$ is 3.

Example 2: If $p(x) = x^2 - \frac{x}{4}$, then find the value of $p(3)$.

Solution:

It is given that $p(x) = x^2 - \frac{x}{4}$.

On replacing x with 3, we get:

$$p(x) = (3)^2 - \frac{3}{4}$$
$$= 9 - \frac{3}{4}$$
$$= \frac{33}{4}$$

Thus, the value of $x^2 - \frac{x}{4}$ at $p(3)$ is $\frac{33}{4}$.

Medium

Example 1: If $p(x) = x(x - 4)$ and $q(y) = (y + 5)(y + 9)$, then show that $p(24) = q(15)$.

Solution:

We have $p(x) = x(x - 4)$

$\therefore p(24) = 24(24 - 4)$

$= 480$ ... (1)

Also, $q(y) = (y + 5)(y + 9)$

$\therefore q(15) = (15 + 5)(15 + 9)$

$= 20 \times 24$

$= 480$ ... (2)

From equations 1 and 2, we get:

$p(24) = q(15)$

Example 2: If $p(a) = (3a - 2)(3a + 2)$ and $q(b) = b(b + 5)^2$, then find the value of $q(25) - p(18)$.

Solution:

We have $p(a) = (3a - 2)(3a + 2)$

$\Rightarrow p(a) = (3a)^2 - 2^2$ [$\because (a - b)(a + b) = a^2 - b^2$]

$\therefore$ **$p(18) = (3 \times 18)^2 - 2^2$**

$= 54^2 - 2^2$

$= 2916 - 4$

$= 2912$

Also, **$q(b) = b(b + 5)^2$**

$\therefore$ **$q(25) = 25\ (25 + 5)^2$**

$= 25 \times 30^2$

$= 25 \times 900$

$= 22500$

Now, $q(25) - p(18) = 22500 - 2912 = 19588$

Hard

Example 1: If $p(x) = x^3 - 5ax^2 + 3bx$ and $p(2) = p(5)$, then prove that $35a - 3b = 39$.

Solution:

We have $p(x) = x^3 - 5ax^2 + 3bx$

$\therefore p(2) = 2^3 - 5 \times a \times 2^2 + 3 \times b \times 2$

$= 8 - 20a + 6b$ … (1)

And $p(5) = 5^3 - 5 \times a \times 5^2 + 3 \times b \times 5$

$= 125 - 125a + 15b$ … (2)

According to the given information:

$p(2) = p(5)$

So, on using equations 1 and 2, we get:

$8 - 20a + 6b = 125 - 125a + 15b$

$\Rightarrow 125a - 20a + 6b - 15b = 125 - 8$

$\Rightarrow 105a - 9b = 117$

$\Rightarrow 3\ (35a - 3b) = 117$

$\Rightarrow \therefore 35a - 3b = 39$

Zeroes of Polynomials

Zero Value of a Polynomial

We know that the value of a polynomial differs according to the value of the variable in it. For example, if $p(x)$ is a polynomial with variable x and we put different values of x in $p(x)$, then we will get different values of $p(x)$. In some cases, the value of $p(x)$ can be the same for two or more values of x. Also, in a few cases, the value of $p(x)$ can be zero.

The values of the variable at which a polynomial becomes zero are called **zeroes of the polynomial**. These values are very special and useful to us. Zeroes of polynomials are used in:

- Solving problems related to **motion**

.Solving problems related to **path** or **Focus** of a **point** or geometric figure

- Making graphs for economic **data**

In this lesson, we will learn to check whether or not the given values are zeroes of the given polynomials. We will also learn to find the zeroes of different polynomials.

Zeroes or Roots of a Polynomial

If the value of a polynomial $p(x)$ at $x = a$ **is zero, then a is said to be the zero or root of the polynomial $p(x)$.** Let us consider the polynomial $p(x) = x^2 - 5x + 6$ and check its values at
$x = 1, 2$ ….

$p(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2$

$p(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

$p(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Clearly, $p(2)$ and $p(3)$ are equal to 0; so, $x = 2$ and $x = 3$ are the zeroes or roots of the given polynomial.

The value of a **constant**

polynomial can never be zero. Hence, a constant polynomial has no zeroes or roots. For example, $p(x) = 8$ is a constant polynomial. Let us try to find the roots of this polynomial.

On replacing x with any number, we will always get 8. Suppose we replace x with 2. Then, $p(2)$ will still be equal to 8. This will be the case for any value of x.

Did You Know?

- The maximum number of roots of a polynomial is less than or equal to the degree of the polynomial. For example, the polynomial $x^3 - 2x + 10$ has a degree 3; so, the number of roots of this polynomial will be 3, 2 or 1.
- Every non-constant polynomial has at least one root.
- A polynomial can have more than one root.

Whiz Kid

The roots of quadratic polynomials of the form $ax^2 + bx + c$ can be found by the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, the root of $x^2 + 2x + 1$ can be found as follows:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm \sqrt{0}}{2} = \frac{-2}{2} = -1$$

Solved Examples

Easy

Example 1: Check if $\frac{5}{2}$ is a zero of the polynomial, $p(x) = 2x^2 + 3x - 20$.

Solution:

We have the polynomial, $\boldsymbol{p(x) = 2x^2 + 3x - 20}$.

On replacing x with $\frac{5}{2}$, we get:

$$p\left(\frac{5}{2}\right)=2\left(\frac{5}{2}\right)^2+3\left(\frac{5}{2}\right)-20$$
$$=\frac{25}{2}+\frac{15}{2}-20$$
$$=20-20$$
$$=0$$

So, the value of $p(x)$ is zero at $x = \frac{5}{2}$. Hence, $\frac{5}{2}$ is a zero of the polynomial.

Example 2: Check if $-k$ is a zero of the polynomial, $p(t) = 2t^3 - 2t(1 - t) + 3k$.

Solution:

We have the polynomial, $\boldsymbol{p(t) = 2t^3 - 2t(1 - t) + 3k}$.

On replacing t with $-k$, we get:

$p(-k) = 2(-k)^3 - 2(-k)[1 - (-k)] + 3k$

$= -2k^3 + 2k(1 + k) + 3k$

$= -2k^3 + 2k + 2k^2 + 3k$

$= -2k^3 + 2k^2 + 2k + 3k$

$= -2k^3 + 2k^2 + 5k$

Clearly, the value of $p(t)$ is not zero at $t = -k$. Hence, $-k$ is not a zero of the polynomial.

Medium

Example 1: Find the value of a for which 3 is the zero of the polynomial, $g(z) = 5z^2 - 7z + 6a$.

Solution:

We have the polynomial, $\boldsymbol{g(z) = 5z^2 - 7z + 6a}$.

It is given that 3 is a zero of $g(z)$. This means that $g(3) = 0$

$\Rightarrow 5 \times 3^2 - 7 \times 3 + 6a = 0$

$\Rightarrow 45 - 21 + 6a = 0$

$\Rightarrow 24 + 6a = 0$

$\Rightarrow 6a = -24$

$\Rightarrow a = -4$

Thus, 3 is a zero of the polynomial $\mathbf{5z^2 - 7z + 6a}$ when $a = -4$.

Example 2:

Find the zeroes of the following polynomials.

i) $p(x) = 4x - 5$

ii) $p(x) = x^2 - 2x$

iii) $p(x) = 3x + 4$

Solution:

i) We have the polynomial, $p(x) = 4x - 5$.

To find the zero, let us put $p(x) = 0$.

$\Rightarrow \mathbf{4x - 5 = 0}$

$\Rightarrow \mathbf{4x = 5}$

$\Rightarrow x = \frac{5}{4}$

Therefore, $\frac{5}{4}$ is the zero of the polynomial $\mathbf{4x - 5}$.

ii) We have the polynomial, $p(x) = x^2 - 2x$

To find the zero, let us put $p(x) = 0$.

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x - 2) = 0$

$\Rightarrow$ We can have $x = 0$ or $x - 2 = 0$.

$\Rightarrow x = 0$ or $x = 2$

Therefore, 0 and 2 are the two zeroes of the polynomial $x^2 - 2x$.

iii) We have the polynomial, $p(x) = 3x + 4$.

To find the zero, let us put $p(x) = 0$.

$\Rightarrow \mathbf{3x + 4 = 0}$

$\Rightarrow \mathbf{3x = -4}$

$\Rightarrow x = \frac{-4}{3}$

Therefore, $-\frac{4}{3}$ is the zero of the polynomial **3x + 4**.

Hard

Example 1: Find the zero of the polynomial, $p(y) = a(y - a) - b(y - b)$, where $a - b \neq 0$.

Solution:

We have the polynomial, $\boldsymbol{p(y) = a(y - a) - b(y - b)}$.

To find the zero, let us put $p(y) = 0$.

$\Rightarrow a(y - a) - b(y - b) = 0$

$\Rightarrow ay - a^2 - by + b^2 = 0$

$\Rightarrow ay - by = a^2 - b^2$

$\Rightarrow y(a - b) = (a - b)(a + b)$

$\Rightarrow y = a + b$

Hence, $a + b$ is the zero of the polynomial $\boldsymbol{a(y - a) - b(y - b)}$.

Example 2: A stone is thrown upward from the ground. After gaining a height of 12 m, the change in height of the stone is given by the relation $10t - 5t^2$, where t represents time in seconds. Calculate the time taken by the stone to return to the ground.

Solution:

The change in the height of the stone is given by the polynomial, $p(t) = 10t - 5t^2$.

The height of the stone becomes zero when it reaches the ground.

Therefore, we have $p(t) = 0$.

$\Rightarrow 10t - 5t^2 = 0$

$\Rightarrow 5t\,(2 - t) = 0$

$\Rightarrow$ We can have $t = 0$ or $2 - t = 0$.

$\Rightarrow t = 0$ or $t = 2$

Time, $t = 0$ when the stone reaches the maximum height of 12 m. So, the time taken by the stone to reach the ground is two seconds.

Division of Polynomials by Polynomials (Degree 1) Using Long Division Method

Dividing one number by another is something that we know well. For example, let us divide 434 by 9.

```
     48
9) 434
  -36
    74
   -72
     2
```

In the above division, 434 is the **dividend**

, 9 is the **divisor**
, 48 is the **quotient**
and 2 is the **remainder**
.

We also know how to represent any division using the division algorithm, which states that:

Dividend = Divisor × Quotient + Remainder

Thus, we can write 434 as:

434 = 9 × 48 + 2

We can divide one polynomial by another in the same way as we divide one number by another. In this lesson, we will learn to carry out the division of polynomials and verify the same using the division algorithm.

Division Algorithm

Polynomials also satisfy the division algorithm.

Consider the division of $2x^2 - 9x + 4$ by $x - 2$.

In this division, we have

Dividend = $2x^2 - 9x + 4$

Divisor = $x - 2$

Quotient = $2x - 5$

Remainder = -6

Now,

Divisor × Quotient + Remainder = $[(x - 2)(2x - 5)] + (-6)$

$= [x(2x - 5) - 2(2x - 5)] - 6$

$= 2x^2 - 5x - 4x + 10 - 6$

$= 2x^2 - 9x + 4$

= Dividend

Thus, the given division satisfies the division algorithm, i.e,

Dividend = Divisor × Quotient + Remainder

Solved Examples

Easy

Example 1: Divide the polynomial by monomial and write the quotient and remainder.

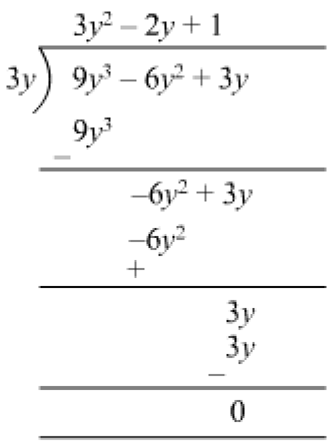
(i) $9y^3 - 6y^2 + 3y \div 3y$

(ii) $8x^4 + 4x^2 - 5x + 1 \div 2x$

(iii) $4z^6 - 18z^4 + 24z^2 \div 12z$

Solution:

(i) $9y^3 - 6y^2 + 3y$ can be divided by $3y$ using long division method as follows:



Therefore,

Quotient = $3y^2 - 2y + 1$

Remainder = 0

(ii) $8x^4 + 4x^2 - 5x + 1$ can be divided by $2x$ using long division method as follows:

$$4x^3 + 2x - \frac{5}{2}$$

$$2x \overline{)\, 8x^4 + 4x^2 - 5x + 1}$$

$$8x^4$$

$$-$$

$$+4x^2 - 5x + 1$$

$$+4x^2$$

$$-$$

$$-5x + 1$$

$$-5x$$

$$+$$

$$1$$

Therefore,

Quotient = $4x^3 + 2x - \frac{5}{2}$

Remainder = 1

(iii) $4z^6 - 18z^4 + 24z^2$ can be divided by $12z$ using long division method as follows:

$$\frac{1}{3}z^5 - \frac{3}{2}z^3 + 2z$$

$$12z \overline{)\, 4z^6 - 18z^4 + 24z^2}$$

$$4z^6$$

$$-$$

$$-18z^4 + 24z^2$$

$$-18z^4$$

$$+$$

$$24z^2$$

$$24z^2$$

$$-$$

$$0$$

Therefore,

Quotient = $\frac{1}{3}z^5 - \frac{3}{2}z^3 + 2z$

Remainder = 0

Example 2: Divide $5x^2 + 7x - 4$ by $x + 1$.

Solution:

Let $p(x) = 5x^2 + 7x - 4$ and $q(x) = x + 1$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & \phantom{5x^2 + {}}5x + 2 \\
\hline
x + 1 & 5x^2 + 7x - 4 \\
 & 5x^2 + 5x \\
 & - \quad\; - \\
\hline
 & \qquad 2x - 4 \\
 & \qquad 2x + 2 \\
 & \qquad - \quad - \\
\hline
 & \qquad\quad -6 \\
\hline
\end{array}
$$

Example 3: What is the quotient when $-2x^2 + x^3 - 9x + 18$ is divided by $x - 2$?

Solution:

Let $p(x) = -2x^2 + x^3 - 9x + 18$

$= x^3 - 2x^2 - 9x + 18$ (arranging the terms in decreasing order of their powers)

Let $q(x) = x - 2$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r l}
 & x^2 - 9 \\
x-2 & \overline{)\,x^3 - 2x^2 - 9x + 18} \\
 & x^3 - 2x^2 \\
 & \underline{-\quad +} \\
 & \qquad -9x + 18 \\
 & \qquad -9x + 18 \\
 & \qquad \underline{+ \quad -} \\
 & \qquad\qquad 0
\end{array}
$$

Hence, when $-2x^2 + x^3 - 9x + 18$ is divided by $x - 2$, the quotient is $x^2 - 9$.

Medium

Example 1: Find the quotient and remainder on dividing $f(x)$ by $g(x)$, where

$f(x) = 6x^3 + 13x^2 + x - 2$ and $g(x) = 2x + 1$.

Solution:

We have $f(x) = 6x^3 + 13x^2 + x - 2$ and $g(x) = 2x + 1$.

The terms of $f(x)$ and $g(x)$ are arranged in decreasing order of their powers. Let us divide $f(x)$ by $g(x)$.

$$
\begin{array}{r l}
 & 3x^2 + 5x - 2 \\
2x+1 & \overline{)\,6x^3 + 13x^2 + x - 2} \\
 & 6x^3 + 3x^2 \\
 & \underline{-\quad -} \\
 & \qquad 10x^2 + x \\
 & \qquad 10x^2 + 5x \\
 & \qquad \underline{-\quad -} \\
 & \qquad\qquad -4x - 2 \\
 & \qquad\qquad -4x - 2 \\
 & \qquad\qquad \underline{+\quad +} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Thus, when $6x^3 + 13x^2 + x - 2$ is divided by $2x + 1$, we get $3x^2 + 5x - 2$ as the quotient and zero as the remainder.

Example 2: What are the quotient and remainder when $x^4 + 2x^3 - 33x^2 + 18x + 28$ is divided by $x + 7$?

Solution:

Let $p(x) = x^4 + 2x^3 - 33x^2 + 18x + 28$ and $q(x) = x$ +https://www.google.co.in/?gws_rd=ssl 7

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^3 - 5x^2 + 2x + 4 \\
x+7 & x^4 + 2x^3 - 33x^2 + 18x + 28 \\
 & x^4 + 7x^3 \\
 & - \quad - \\
\hline
 & -5x^3 - 33x^2 + 18x + 28 \\
 & -5x^3 - 35x^2 \\
 & + \quad + \\
\hline
 & 2x^2 + 18x + 28 \\
 & 2x^2 + 14x \\
 & - \quad - \\
\hline
 & 4x + 28 \\
 & 4x + 28 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$$\left(\frac{x^4}{x} = x^3\right)$$

$$\left(-\frac{5x^3}{x} = -5x^2\right)$$

$$\left(\frac{2x^2}{x} = 2x\right)$$

$$\left(\frac{4x}{x} = 4\right)$$

Thus, when $x^4 + 2x^3 - 33x^2 + 18x + 28$ is divided by $x + 7$, we get $x^3 - 5x^2 + 2x + 4$ as the quotient and zero as the remainder.

Hard

Example 1: If the division of $x^3 + 2x^2 + kx + 3$ by $x - 3$ yields 21 as the remainder, then find the quotient and the value of k.

Solution:

Let $p(x) = x^3 + 2x^2 + kx + 3$ and $q(x) = x - 3$. It is given that the division of $p(x)$ by $q(x)$ gives 21 as the remainder.

According to the division algorithm:

Dividend = Divisor × Quotient + Remainder

$\Rightarrow x^3 + 2x^2 + kx + 3 = (x - 3) \times \text{Quotient} + 21$

$\Rightarrow x^3 + 2x^2 + kx + 3 - 21 = (x - 3) \times \text{Quotient}$

$$\therefore \text{Quotient} = \frac{x^3 + 2x^2 + kx - 18}{x - 3}$$

$$\begin{array}{r|l} & x^2 + 5x + (k+15) \\ \hline x - 3 & x^3 + 2x^2 + kx - 18 \\ & x^3 - 3x^2 \\ & - \quad + \\ \hline & 5x^2 + kx \\ & 5x^2 - 15x \\ & - \quad + \\ \hline & (k+15)x - 18 \\ & (k+15)x - 3k - 45 \\ & - \quad\quad + \quad + \\ \hline & 3k + 27 \\ \hline \end{array}$$

Now, $3k + 27 = 0$

$\Rightarrow 3k = -27$

$\Rightarrow k = -9$

So, quotient $= x^2 + 5x + (k + 15)$

$= x^2 + 5x + (-9 + 15)$

$= x^2 + 5x + 6$

Example 2: Divide $3x^3 + 5x^2 + 4x + 7$ by $3x - 1$. Find the quotient and the remainder, and verify using the division algorithm.

Solution:

Let $p(x) = 3x^3 + 5x^2 + 4x + 7$ and $q(x) = 3x - 1$

The terms of $p(x)$ and $q(x)$ are arranged in decreasing order of their powers. Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^2 + 2x + 2 \\
\hline
3x - 1 & 3x^3 + 5x^2 + 4x + 7 \\
 & 3x^3 - x^2 \\
 & - \quad + \\
\hline
 & \quad 6x^2 + 4x \\
 & \quad 6x^2 - 2x \\
 & \quad - \quad + \\
\hline
 & \qquad 6x + 7 \\
 & \qquad 6x - 2 \\
 & \qquad - \quad + \\
\hline
 & \qquad\qquad 9 \\
\hline
\end{array}
$$

We have obtained the quotient as $x^2 + 2x + 2$ and the remainder as 9.

Let us verify our result using the division algorithm, which states that:

Dividend = Divisor × Quotient + Remainder

Now,

Divisor × Quotient + Remainder = $[(3x - 1)(x^2 + 2x + 2)] + 9$

$= [3x(x^2 + 2x + 2) - 1(x^2 + 2x + 2)] + 9$

$= 3x^3 + 6x^2 + 6x - x^2 - 2x - 2 + 9$

$= 3x^3 + 5x^2 + 4x + 7$

= Dividend

Thus, the division satisfies the division algorithm.

Remainder Theorem and Its Application

Remainder Theorem

Consider two polynomials $p(x)$ and $q(x)$, where $p(x) = 5x^4 - 4x^2 - 50$ and $q(x) = x - 2$. We know how to divide $p(x)$ by $q(x)$ using the long division method. The result of this division will give the quotient as $5x^3 + 10x^2 + 16x + 32$ and the remainder as 14.

The long division method of finding the remainder is quite tedious. There is a simpler way to find the above remainder. This method is generalized in the form of a theorem called the **remainder theorem**. This theorem helps us find the remainder when a polynomial is to be divided by a linear polynomial.

In this lesson, we will study the remainder theorem and some of its applications in the form of examples.

Understanding the Remainder Theorem

Consider the division of a polynomial $p(x)$ by a polynomial $q(x)$, where $p(x) = 5x^4 - 4x^2 - 50$ and $q(x) = x - 2$. In this case, we have:

Dividend = $p(x)$ and divisor = $q(x)$

On dividing $p(x)$ by $q(x)$ using the long division method, we get:

Quotient = $5x^3 + 10x^2 + 16x + 32$ and remainder = 14

Now, let us find the value of $p(x)$ at $x = 2$.

$p(2) = 5 \times 2^4 - 4 \times 2^2 - 50$

$= 5 \times 16 - 4 \times 4 - 50$

$= 80 - 16 - 50$

$= 14$

Note how the value of $p(2)$ is the same as the remainder obtained by the long division of $p(x)$ by $q(x)$. Also observe how $x = 2$ is a zero of the polynomial $q(x)$.

Thus, **if we replace x in the dividend with the zero (or root) of the divisor, then we get the remainder**.

This method of finding the remainder is called the remainder theorem. It can be stated as follows:

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number

***a*, if *p*(*x*) is divided by a linear polynomial *x* − *a*, then the remainder will be *p*(*a*).**

Proof of the Remainder Theorem

Statement

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(x)$ is divided by a linear polynomial $x - a$, then the remainder will be $p(a)$.

Proof

Let $p(x)$ be a polynomial of a degree greater than or equal to 1 and a be any real number. When divided by $x - a$, let $p(x)$ leave the remainder $r(x)$. Let $q(x)$ be the quotient obtained.

Then, $p(x) = (x - a)\, q(x) + r(x)$, where $r(x) = 0$ or degree $r(x)$ < degree $(x - a)$

Now, $x - a$ is a polynomial of degree 1; so, either $r(x) = 0$ or $r(x)$ = constant (since a polynomial of degree less than 1 is a constant).

Let $r(x)$ = constant = r (say). Then, $p(x) = (x - a)\, q(x) + r$

On putting $x = a$, we get $p(a) = (a - a)\, q(a) + r = 0 \times q(a) + r = r$

Thus, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

Notes:

1) If $p(x)$ is divided by $x + a$, then the remainder will be $p(-a)$.

2) If $p(x)$ is divided by $ax - b$, then the remainder will be $p\left(\frac{b}{a}\right)$.

3) If $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

Solved Examples

Easy

Example 1: Find the remainder when $x^3 - x^2a + 5xa$ is divided by $x - a$.

Solution: Let $p(x) = x^3 - x^2a + 5xa$

According to the remainder theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - a = 0$, we get $x = a$.

∴ Remainder = $p(a)$

$= a^3 - a^2 \times a + 5 \times a \times a$

$= a^3 - a^3 + 5a^2$

$= 5a^2$

Thus, when $x^3 - x^2a + 5xa$ is divided by $x - a$, we get $5a^2$ as the remainder.

Example 2: What is the remainder when $81x^4 + 54x^3 - 9x^2 - 3x + 2$ is divided by $3x + 2$?

Solution:

Let $p(x) = 81x^4 + 54x^3 - 9x^2 - 3x + 2$

As per the remainder theorem, if $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

On putting $3x + 2 = 0$, we get $x = -\frac{2}{3}$.

$$\therefore \text{Remainder} = p\left(-\frac{2}{3}\right)$$

$$= 81\left(-\frac{2}{3}\right)^4 + 54\left(-\frac{2}{3}\right)^3 - 9\left(-\frac{2}{3}\right)^2 - 3\left(-\frac{2}{3}\right) + 2$$

$$= 81 \times \frac{16}{81} - 54 \times \frac{8}{27} - 9 \times \frac{4}{9} + 3 \times \frac{2}{3} + 2$$

$$= 16 - 16 - 4 + 2 + 2$$

$$= 0$$

Thus, when $81x^4 + 54x^3 - 9x^2 - 3x + 2$ is divided by $3x + 2$, we get zero as the remainder.

Medium

Example 1: Verify the remainder theorem for the division of $2x^3 - 3x^2 + 4$ by $x - 3$.

Solution:

Let $p(x) = 2x^3 - 3x^2 + 4$

Let us divide $p(x)$ by $x - 3$ using the long division method.

$$
\begin{array}{r|l}
 & 2x^2 + 3x + 9 \\
\hline
x - 3 & 2x^3 - 3x^2 + 4 \\
 & 2x^3 - 6x^2 \\
 & - \quad + \\
\hline
 & \quad 3x^2 + 4 \\
 & \quad 3x^2 - 9x \\
 & \quad - \quad + \\
\hline
 & \quad\quad 9x + 4 \\
 & \quad\quad 9x - 27 \\
 & \quad\quad - \quad + \\
\hline
 & \quad\quad\quad 31 \\
\hline
\end{array}
$$

Thus, the division of $2x^3 - 3x^2 + 4$ by $x - 3$ yields the remainder 31.

Let us now find the remainder using the remainder theorem. According to this theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - 3 = 0$, we get $x = 3$.

∴ Remainder = $p(3)$

$= 2 \times 3^3 - 3 \times 3^2 + 4$

$= 54 - 27 + 4$

$= 31$

Clearly, the remainder obtained by using the remainder theorem is the same as that obtained via the long division method. Hence, the remainder theorem is verified.

Example 2: For what value of m is $p(x) = mx^3 + 17x^2 - 31x - 2m$ completely divisible by $3x + 1$.

Solution:

It is given that $p(x) = mx^3 + 17x^2 - 31x - 2m$

If $p(x)$ is completely divisible by $ax + b$, then the remainder will be zero, i.e., $p\left(-\frac{b}{a}\right) = 0$.

On putting $3x + 1 = 0$, we get $x = -\frac{1}{3}$.

Using the remainder theorem, we can find the value of m as follows:

$$p\left(-\frac{1}{3}\right) = 0$$
$$\Rightarrow m\left(-\frac{1}{3}\right)^3 + 17\left(-\frac{1}{3}\right)^2 - 31\left(-\frac{1}{3}\right) - 2m = 0$$
$$\Rightarrow m\left(-\frac{1}{27}\right) + 17 \times \frac{1}{9} + \frac{31}{3} - 2m = 0$$
$$\Rightarrow -\frac{m}{27} - 2m + \frac{17}{9} + \frac{31}{3} = 0$$
$$\Rightarrow \frac{-m - 54m}{27} + \frac{17 + 93}{9} = 0$$
$$\Rightarrow -\frac{55m}{27} + \frac{110}{9} = 0$$
$$\Rightarrow -\frac{55m}{27} = -\frac{110}{9}$$
$$\Rightarrow m = -\frac{110}{9} \times \left(-\frac{27}{55}\right)$$

$\Rightarrow m = 6$

Thus, when $m = 6$, $mx^3 + 17x^2 - 31x - 2m$ is completely divisible by $3x + 1$.

Hard

Example 1:

Find the value of k for which $p(x) = 4kx^3 - 13x - 3k + 2$

i) is exactly divisible by $2x - 1$.

ii) leaves 3 as the remainder when divided by $2x + 3$.

Solution:

i)We have $p(x) = 4kx^3 - 13x - 3k + 2$

As per the remainder theorem, if $p(x)$ is divided by $ax - b$, then the remainder will be $p\left(\frac{b}{a}\right)$.

On putting $2x - 1 = 0$, we get $x = \frac{1}{2}$.

Now, if $4kx^3 - 13x - 3k + 2$ is exactly divisible by $2x - 1$, then the remainder will be zero, i.e., $p\left(\frac{1}{2}\right) = 0$.

Using the remainder theorem, we can find the value of k as follows:

$$p\left(\frac{1}{2}\right) = 0$$
$$\Rightarrow 4k\left(\frac{1}{2}\right)^3 - 13\left(\frac{1}{2}\right) - 3k + 2 = 0$$
$$\Rightarrow 4k \times \frac{1}{8} - \frac{13}{2} - 3k + 2 = 0$$
$$\Rightarrow \frac{k}{2} - \frac{13}{2} - 3k + 2 = 0$$
$$\Rightarrow -\frac{5k}{2} - \frac{9}{2} = 0$$
$$\Rightarrow -\frac{5k}{2} = \frac{9}{2}$$

$$\Rightarrow k = \frac{-9}{5}$$

Thus, when $k = \frac{-9}{5}$, $4kx^3 - 13x - 3k + 2$ is exactly divisible by $2x - 1$.

ii)As per the remainder theorem, if $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

On putting $2x + 3 = 0$, we get $x = -\frac{3}{2}$.

It is given that the division of $4kx^3 - 13x - 3k + 2$ by $2x + 3$ yields the remainder 3,

i.e., $p\left(-\frac{3}{2}\right)=3$.

Using the remainder theorem, we can find the value of k as follows:

$$p\left(-\frac{3}{2}\right)=3$$

$$\Rightarrow 4k\left(-\frac{3}{2}\right)^3-13\left(-\frac{3}{2}\right)-3k+2=3$$

$$\Rightarrow 4k\times\left(-\frac{27}{8}\right)+\frac{39}{2}-3k+2=3$$

$$\Rightarrow \frac{-27k}{2}+\frac{39}{2}-3k+2=3$$

$$\Rightarrow -\frac{33}{2}k+\frac{43}{2}=3$$

$$\Rightarrow -\frac{33}{2}k=3-\frac{43}{2}$$

$$\Rightarrow -\frac{33}{2}k=-\frac{37}{2}$$

$$\Rightarrow k=\frac{37}{33}$$

Thus, when $k = \frac{37}{33}$, the division of $4kx^3 - 13x - 3k + 2$ by $2x + 3$ leaves 3 as the remainder.

Example 2: Find the values of a and b for which $p(x) = x^3 + ax^2 + bx - 20$ leaves 0 and −2 as the remainders when divided by $x - 5$ and $x - 3$ respectively.

Solution:

We have $p(x) = x^3 + ax^2 + bx - 20$

As per the remainder theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - 5 = 0$, we get $x = 5$.

On putting $x - 3 = 0$, we get $x = 3$.

Now, if the division of $x^3 + ax^2 + bx - 20$ by $x - 5$ leaves 0 as the remainder, then $p(5) = 0$.

$\Rightarrow 5^3 + a \times 5^2 + b \times 5 - 20 = 0$
$\Rightarrow 125 + 25a + 5b - 20 = 0$
$\Rightarrow 25a + 5b + 105 = 0$
$\Rightarrow 5a + b = -21 \quad ...(1)$

Also, if the division of $x^3 + ax^2 + bx - 20$ by $x - 3$ leaves −2 as the remainder, then $p(3) = -2$.

$\Rightarrow 3^3 + a \times 3^2 + b \times 3 - 20 = -2$
$\Rightarrow 27 + 9a + 3b = -2 + 20$
$\Rightarrow 9a + 3b = 18 - 27$
$\Rightarrow 9a + 3b = -9$
$\Rightarrow 3a + b = -3 \quad ...(2)$

On solving equations 1 and 2, we get:

$5a + (-3 - 3a) = -21$ ($\because b = -3 - 3a$)

$\Rightarrow 5a - 3a - 3 = -21$

$\Rightarrow 2a = -21 + 3$

$\Rightarrow 2a = -18$

$\Rightarrow a = -9$

Now, $b = -3 - 3a$

$\Rightarrow b = -3 - 3 \times (-9)$

$\Rightarrow b = -3 + 27$

$\Rightarrow b = 24$

Thus, when $a = -9$ and $b = 24$, the divisions of $x^3 + ax^2 + bx - 20$ by $x - 5$ and $x - 3$ leave 0 and −2 respectively as the remainders.

Factor Theorem and Its Applications

Factor Theorem

We know the relation between a number and its factor. If we divide 91 by 7, then we get 13 as the quotient and zero as the remainder. In this case, we say that 7 is a factor of 91 as the remainder is zero. Now, if we divide 107 by 9, then we get 11 as the quotient and 8 as the remainder. In this case, we say that 9 is not a factor of 107 as the remainder is not zero.

Thus, the relation between a number and its factor is given as follows:

If a number is completely divisible by another number, i.e., the remainder is zero, then the second number is a factor of the first number.

Similarly, a polynomial $p(x)$ is said to be completely divisible by a polynomial $q(x)$ if we get zero as the remainder on dividing $p(x)$ by $q(x)$. In this case, we say that $q(x)$ is a factor of $p(x)$.

We have studied the remainder **theorem** that helps us to find the remainder. Similarly, we have a **factor theorem** that helps us to determine whether or not a polynomial is a factor of another polynomial, without actually performing the division.

In this lesson, we will study the factor theorem and solve some problems based on it.

Understanding the Factor Theorem

We can easily determine whether a polynomial $q(x)$ is a factor of a polynomial $p(x)$ without performing the division. This can be done by using the factor theorem, which can be stated as follows:

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number c,

i) if $p(c) = 0$, then $x - c$ will be a factor of $p(x)$ and

ii) if $x - c$ is a factor of $p(x)$, then $p(c)$ will be equal to zero.

Consider the polynomial, $p(x) = x^2 - 3x + 2$.

On putting $x = 2$ in $p(x)$, we get:

$p(2) = 2^2 - 3 \times 2 + 2$

$= 4 - 6 + 2$

$= 0$

Thus, we can say that $x - 2$ is a factor of $p(x)$, where 2 is a real number.

Proof of the Factor Theorem

Statement

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number c,

i) if $p(c) = 0$, then $x - c$ will be a factor of $p(x)$ and

ii) if $x - c$ is a factor of $p(x)$, then $p(c)$ will be equal to zero.

Proof

Let $p(x)$ be a polynomial of a degree greater than or equal to 1 and c be any real number such that $p(c) = 0$. Let quotient $q(x)$ be obtained when $p(x)$ is divided by $x - c$.

i) $\underline{p(c) = 0}$

By the remainder theorem, the remainder obtained is $p(c)$.

$\Rightarrow p(x) = (x - c)\, q(x) + p(c)$

$\Rightarrow p(x) = (x - c)\, q(x)$ [$\because p(c) = 0$]

$\Rightarrow x - c$ is a factor of $p(x)$.

ii) $x - c$ is a factor of $p(x)$

$\Rightarrow$ When divided by $x - c$, $p(x)$ leaves zero as the remainder.

However, by the remainder theorem, the remainder obtained is $p(c)$.

$\Rightarrow p(c) = 0$

Notes

1) $x + c$ will be a factor of $p(x)$ if $p(-c) = 0$

2) $cx - d$ will be a factor of $p(x)$ if $p\left(\frac{d}{c}\right) = 0$

3) $cx + d$ will be a factor of $p(x)$ if $p\left(-\frac{d}{c}\right) = 0$

4) $(x - c)(x - d)$ will be a factor of $p(x)$ if $p(c) = 0$ and $p(d) = 0$

Solved Examples

Easy

Example 1: Check whether or not $x - 1$ is a factor of $x^3 - 2x^2 - x + 2$.

Solution:

Let $p(x) = x^3 - 2x^2 - x + 2$

According to the factor theorem, $x - 1$ will be a factor of $p(x)$ if $p(1) = 0$.

$p(1) = 1^3 - 2 \times 1^2 - 1 + 2$

$= 1 - 2 - 1 + 2$

$= 0$

Thus, $x - 1$ is a factor of $x^3 - 2x^2 - x + 2$.

Example 2: Using the factor theorem, show that $2x + 1$ is a factor of $2x^3 + 3x^2 - 11x - 6$.

Solution:

Let $p(x) = 2x^3 + 3x^2 - 11x - 6$

According to the factor theorem, $2x + 1$ will be a factor of $p(x)$ if $p\left(-\frac{1}{2}\right) = 0$.

$$p\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 11\left(-\frac{1}{2}\right) - 6$$
$$= -\frac{1}{4} + \frac{3}{4} + \frac{11}{2} - 6$$
$$= \frac{-1 + 3 + 22 - 24}{4}$$
$$= 0$$

Thus, $2x + 1$ is a factor of $2x^3 + 3x^2 - 11x - 6$.

Medium

Example 1: For what value of m is $x - 3$ a factor of $3x^2 - 3x + m$?

Solution:

Let $p(x) = 3x^2 - 3x + m$

According to the factor theorem, $x - 3$ will be a factor of $p(x)$ if $p(3) = 0$.

$p(3) = 3 \times 3^2 - 3 \times 3 + m$

So, $3 \times 3^2 - 3 \times 3 + m = 0$

$\Rightarrow 3 \times 9 - 9 + m = 0$

$\Rightarrow 27 - 9 + m = 0$

$\Rightarrow 18 + m = 0$

$\Rightarrow m = -18$

Thus, $x - 3$ is a factor of $3x^2 - 3x + m$ when $m = -18$.

Example 2: Check whether or not $2x^2 - 11x + 25$ is exactly divisible by $2x - 3$.

Solution:

Let $p(x) = 2x^2 - 11x + 25$ and $q(x) = 2x - 3$

We know that $p(x)$ will be exactly divisible by $q(x)$ if $q(x)$ is a factor of $p(x)$.

On putting $2x - 3 = 0$, we get $x = \frac{3}{2}$.

On using the factor theorem, we get:

$$\begin{aligned} p\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^2 - 11\left(\frac{3}{2}\right) + 25 \\ &= \frac{9}{2} - \frac{33}{2} + 25 \\ &= 13 \\ &\neq 0 \end{aligned}$$

Thus, $q(x)$ is not a factor of $p(x)$.

Hence, $2x^2 - 11x + 25$ is not exactly divisible by $2x - 3$.

Example 3: Using the factor theorem, determine whether or not $g(x)$ is a factor of $f(x)$, where

$f(x) = 7x^2 - 2\sqrt{8}\,x - 6$ and $g(x) = x - \sqrt{2}$.

Solution:

It is given that $f(x) = 7x^2 - 2\sqrt{8}\,x - 6$ and $g(x) = x - \sqrt{2}$

According to the factor theorem, $g(x)$ will be a factor of $f(x)$ if $f\left(\sqrt{2}\right) = 0$.

$f\left(\sqrt{2}\right) = 7(\sqrt{2})^2 - 2\sqrt{8} \times \sqrt{2} - 6$

$= 7 \times 2 - 2\sqrt{16} - 6$

$= 14 - 8 - 6$

$= 0$

Therefore, $g(x)$ is a factor of $f(x)$.

Hard

Example 1: Using the factor theorem, show that $a-b$, $b-c$ and $c-a$ are factors of $a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$.

Solution:

We have the given expression as $a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$.

As per the factor theorem, $x - k$ will be a factor of a polynomial $p(x)$ if $p(x) = 0$ when $x = k$.

Let us consider $p(a) = a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$ to be a polynomial in variable 'a'. Take b and c as constants for the time being.

Now, as per the factor theorem, $a - b$ will be a factor of $p(a)$ if $p(a) = 0$ when $a = b$.

On putting $a = b$ in $p(a)$, we get:

$b\ (b^2 - c^2) + b\ (c^2 - b^2) + c\ (b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c \times 0$

$= 0$

Thus, $a - b$ is a factor of $a\ (b^2 - c^2) + b\ (c^2 - a^2) + c\ (a^2 - b^2)$.

Now, suppose $p(b) = a\ (b^2 - c^2) + b\ (c^2 - a^2) + c\ (a^2 - b^2)$ is a polynomial in variable 'b' and a and c are constants. Then, $b - c$ will be a factor of $p(b)$ if $p(b) = 0$ when $b = c$.

On substituting $b = c$ in $p(b)$, we find that the result is zero.

Similarly, we can take $p(c) = a\ (b^2 - c^2) + b\ (c^2 - a^2) + c\ (a^2 - b^2)$ to be a polynomial in variable 'c' and a and b as constants. Then, $c - a$ will be a factor of $p(c)$ if $p(c) = 0$ when $c = a$. On substituting $c = a$ in $p(c)$, we find that the result is zero.

Hence, $b - c$ and $c - a$ are also factors of $a\ (b^2 - c^2) + b\ (c^2 - a^2) + c\ (a^2 - b^2)$.

Factorisation of Quadratic Polynomials Using Factor Theorem and Splitting Middle Term

Factorisation of Quadratic Polynomials

We know that 7 × 6 = 42. Here, 7 and 6 are factors of 42. Now, consider the linear polynomials
$x - 2$ and $x + 1$. On multiplying the two, we get: $x\ (x + 1) - 2\ (x + 1) = x^2 + x - 2x - 2 = x^2 - x - 2$, which is a quadratic polynomial. So, $x - 2$ and $x + 1$ are factors of the quadratic polynomial
$x^2 - x - 2$. A quadratic polynomial can have a maximum of two factors.

In the above example, we found the quadratic polynomial from its two factors. We can also find the factors from the quadratic polynomial. This process of decomposing a polynomial into a product of its factors (which when multiplied give the original expression) is called **factorisation**.

There are two ways of finding the factors of quadratic polynomials viz., by applying the factor theorem and by splitting the middle term. We will discuss these methods of factorisation in this lesson and also solve some examples based on them.

Factorisation of Quadratic Polynomials Using the Factor Theorem

The factor theorem states that: **For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(a) = 0$, then $x - a$ will be a factor of $p(x)$.**

Consider the quadratic polynomial, $p(x) = x^2 - 5x + 6$. To find its factors, we need to ascertain the value of x for which the value of the polynomial comes out to be zero. For this, we first determine the factors of the constant term in the polynomial, and then check the value of the polynomial at these points.

In the given polynomial, the constant term is 6 and its factors are ±1, ±2, ±3 and ±6.

Let us now check the value of the polynomial for each of these factors of 6.

$p(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Hence, $x - 3$ is also a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum two factors which are already obtained as: $(x - 2)$ and $(x - 3)$.

Thus, the given polynomial = $p(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$

Solved Examples

Easy

Example 1: Factorise $x^2 - 7x + 10$ using the factor theorem.

Solution:

Let $p(x) = x^2 - 7x + 10$

The constant term is 10 and its factors are ±1, ±2, ±5 and ±10.

Let us check the value of the polynomial for each of these factors of 10.

$p(1) = 1^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(5) = 5^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$

Hence, $x - 5$ is a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.

Thus, we can write the given polynomial as:

$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$

Hard

Example 1: Factorise $x^4y^2 - 5x^2y^2 + 6y^2$.

Solution:

$x^4y^2 - 5x^2y^2 + 6y^2 = y^2(x^4 - 5x^2 + 6)$

Let $x^2 = a$

$\Rightarrow (x^2)^2 = a^2$

$\Rightarrow x^4 = a^2$

$\therefore x^4y^2 - 5x^2y^2 + 6y^2 = y^2(a^2 - 5a + 6)$

$= y^2 \times f(a)$, where $f(a) = a^2 - 5a + 6$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term '6' are ±1, ±2, ±3 and ±6.

$f(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Thus, $a - 2$ is a factor of $f(a)$.

$f(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Thus, $a - 3$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a - 2$ and $a - 3$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 5a + 6 = (a - 2)(a - 3)$

Hence, $x^4y^2 - 5x^2y^2 + 6y^2 = y^2 (a - 2)(a - 3)$

$= y^2 (x^2 - 2)(x^2 - 3)$

Example 2: Factorise $4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2 (y^2 - 2) - 20$.

Solution:

$4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2 (y^2 - 2) - 20$

$= 4xy^2 + 4x^2 - 4x + 12 + (y^2)^2 - 2y^2 - 20$

$= (2x)^2 + (y^2)^2 + 2 \times 2x \times y^2 - 4x - 2y^2 + 12 - 20$

$= (2x + y^2)^2 - 2(2x + y^2) - 8$

$= a^2 - 2a - 8$

$= f(a)$, where $a = 2x + y^2$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term '8' are ±1, ±2, ±4 and ±8.

$f(1) = 1^2 - 2 \times 1 - 8 = 1 - 2 - 8 = -9 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(-1) = (-1)^2 - 2 \times (-1) - 8 = 1 + 2 - 8 = -5 \neq 0$

Thus, $a + 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 2 \times 2 - 8 = 4 - 4 - 8 = -8 \neq 0$

Thus, $a - 2$ is not a factor of $f(a)$.

$f(-2) = (-2)^2 - 2 \times (-2) - 8 = 4 + 4 - 8 = 0$

Thus, $a + 2$ is a factor of $f(a)$.

$f(4) = 4^2 - 2 \times 4 - 8 = 16 - 8 - 8 = 0$

Thus, $a - 4$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a + 2$ and $a - 4$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 2a - 8 = (a + 2)(a - 4)$

Hence, $4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2(y^2 - 2) - 20 = (2x + y^2 + 2)(2x + y^2 - 4)$

Solved Examples

Easy

Example 1: Factorise $12x^2 - \sqrt{2}x - 12$ by splitting the middle term.

Solution:

The given polynomial is $12x^2 - \sqrt{2}x - 12$.

Here, $ac = 12 \times (-12) = -144$. The middle term is $-\sqrt{2}$.

Therefore, we have to split $-\sqrt{2}$ into two numbers such that their product is −144 and their sum is $-\sqrt{2}$.

These numbers are $-9\sqrt{2}$ and $8\sqrt{2}$ ($\because -9\sqrt{2} + 8\sqrt{2} = -\sqrt{2}$ and $-9\sqrt{2} \times 8\sqrt{2} = -144$).

Thus, we have:

$12x^2 - \sqrt{2}x - 12 = 12x^2 - 9\sqrt{2}x + 8\sqrt{2}x - 12$

$= 3\sqrt{2}x(2\sqrt{2}x - 3) + 4(2\sqrt{2}x - 3)$

$= (2\sqrt{2}x - 3)(3\sqrt{2}x + 4)$

Example 2: Factorise $2x^2 - 11x + 15$ by splitting the middle term.

Solution:

The given polynomial is $2x^2 - 11x + 15$.

Here, $ac = 2 \times 15 = 30$. The middle term is -11. Therefore, we have to split -11 into two numbers such that their product is 30 and their sum is -11. These numbers are -5 and -6 [$\because (-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].

Thus, we have:

$2x^2 - 11x + 15 = 2x^2 - 5x - 6x + 15$

$= x(2x - 5) - 3(2x - 5)$

$= (2x - 5)(x - 3)$

Medium

Example 1: Factorise $(3y - 1)^2 - 6y + 2$.

Solution:

$(3y - 1)^2 - 6y + 2 = 9y^2 + 1 - 6y - 6y + 2$

$= 9y^2 - 12y + 3$

$= 3(3y^2 - 4y + 1)$

Here, $ac = 1 \times 3 = 3$. The middle term is -4. Therefore, we have to split -4 into two numbers such that their product is 3 and their sum is -4. These numbers are -1 and -3 [$\because (-3) + (-1) = -4$ and $(-3) \times (-1) = 3$].

Thus, we have:

$3(3y^2 - 4y + 1) = 3(3y^2 - 3y - y + 1)$

$= 3\ [3y\ (y - 1) - 1\ (y - 1)]$

$= 3\ (y - 1)\ (3y - 1)$

Example 2: Find the dimensions of a rectangle whose area is given by the polynomial $20p^2 + 69p + 54$.

Solution:

We know that area of a rectangle = Length × Breadth

Area of the rectangle is given by the polynomial $20p^2 + 69p + 54$. So, its factors will be the required dimensions of the rectangle.

In the given polynomial, $ac = 20 \times 54 = 1080$. The middle term is 69. Therefore, we have to split 69 into two numbers such that their product is 1080 and their sum is 69. These numbers are 45 and 24 ($\because$ $45 + 24 = 69$ and $45 \times 24 = 1080$).

Thus, we have:

$20p^2 + 69p + 54 = 20p^2 + 45p + 24p + 54$

$= 5p\ (4p + 9) + 6\ (4p + 9)$

$= (4p + 9)\ (5p + 6)$

Hence, the dimensions of the rectangle are $5p + 6$ and $4p + 9$.

Hard

Example 1: **Factorise** $2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$.

Solution:

$$2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$$

$$=2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}\right)+9$$

$$=2\left(3x+\frac{4}{5x}\right)^2+18\left(3x+\frac{4}{5x}\right)+\left(3x+\frac{4}{5x}\right)+9$$

$$=2\left(3x+\frac{4}{5x}\right)\left[\left(3x+\frac{4}{5x}\right)+9\right]+1\left[\left(3x+\frac{4}{5x}\right)+9\right]$$

$$=\left[\left(3x+\frac{4}{5x}\right)+9\right]\left[2\left(3x+\frac{4}{5x}\right)+1\right]$$

$$=\left(3x+\frac{4}{5x}+9\right)\left(6x+\frac{8}{5x}+1\right)$$

Factorisation of Cubic Polynomial Using Factor Theorem

Factorization of Cubic Polynomials

A cubic polynomial can be written as $p(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers. We cannot factorize a cubic polynomial in the manner in which we factorize a quadratic polynomial. We use a different approach for this purpose.

A cubic polynomial can have a maximum of three linear factors. By knowing one of these factors, we can reduce it to a quadratic polynomial. Thus, to factorize a cubic polynomial, we first find a factor by the hit and trial method or by using the factor theorem, and then reduce the cubic polynomial into a quadratic polynomial. The resultant quadratic polynomial is solved by splitting its middle term or by using the factor theorem.

In this lesson, we will learn how to factorize a cubic polynomial and solve some examples related to the same.

Know More

Hit and trial method

Hit and trial method is used to find the factors or roots of a polynomial of degree more than two.

In this method, we put some value in the given polynomial to see if it satisfies the polynomial. If it does, then it is the zero of that polynomial. Using this method, we can reduce a polynomial of degree, say n, to a polynomial of degree $n - 1$.

Solved Examples

Easy

Example 1: Factorize $x^3 - 3x^2 - x + 3$.

Solution:

Let $p(x) = x^3 - 3x^2 - x + 3$

The constant term is 3.

The factors of 3 are ±1 and ±3.

Let us take $x = 1$ and find the value of $p(x)$.

$p(1) = 1^3 - 3 \times 1^2 - 1 + 3$

$= 1 - 3 - 1 + 3$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$, using factor theorem.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x) = x^3 - 3x^2 - x + 3$

$= x^3 - x^2 - 2x^2 + 2x - 3x + 3$

$= x^2 (x - 1) - 2x (x - 1) - 3 (x - 1)$

$= (x - 1) (x^2 - 2x - 3)$ … (1)

Next, we factorize $x^2 - 2x - 3$ by splitting its middle term.

The middle term is −2. We have to find two numbers such that their product is −3 and their sum is 2. These two numbers are 3 and −1.

Thus, we have:

$x^2 - 2x - 3 = x^2 - (3 - 1)x - 3$

$= x^2 - 3x + x - 3$

$= x\,(x - 3) + 1\,(x - 3)$

$= (x - 3)\,(x + 1)$

On substituting in equation 1, we get:

$p(x) = (x - 1)\,(x - 3)\,(\,x + 1)$

Example 2: If $x + 3$ is a factor of the polynomial $f(x) = x^3 - 7x + 6$, then factorize $f(x)$.

Solution:

We have $x + 3$ as a factor of the polynomial $f(x) = x^3 + 0x^2 - 7x + 6$.

Let us divide $f(x)$ by $x + 3$.

$$
\begin{array}{r|l}
 & x^2 - 3x + 2 \\
\hline
x + 3 & x^3 + 0x^2 - 7x + 6 \\
 & x^3 + 3x^2 \\
 & - \quad - \\
\hline
 & -3x^2 - 7x \\
 & -3x^2 - 9x \\
 & + \quad + \\
\hline
 & 2x + 6 \\
 & 2x + 6 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$\therefore f(x) = x^3 - 7x + 6 = (x + 3)\,(x^2 - 3x + 2)$

$= (x + 3)\,(x^2 - x - 2x + 2)$

$= (x + 3)\,[x\,(x - 1) - 2\,(x - 1)]$

$= (x + 3)\,(x - 1)\,(x - 2)$

Medium

Example 1: Factorize $2x^3 - 7x^2 + 7x - 2$.

Solution:

Let $p(x) = 2x^3 - 7x^2 + 7x - 2$

Let us take $x = 1$ and find the value of $p(x)$.

$p(1) = 2 \times 1^3 - 7 \times 1^2 + 7 \times 1 - 2$

$= 2 - 7 + 7 - 2$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x) = 2x^3 - 2x^2 - 5x^2 + 5x + 2x - 2$

$= 2x^2 (x - 1) - 5x (x - 1) + 2 (x - 1)$

$=(x - 1) (2x^2 - 5x + 2)$ ... (1)

Next, we factorize $2x^2 - 5x + 2$ by splitting its middle term. The middle term is −5. We have to find two numbers such that their product is 4 and their sum is 5. These two numbers are 4 and 1.

Thus, we have:

$2x^2 - 5x + 2 = 2x^2 - (4 + 1)x + 2$

$= 2x^2 - 4x - x + 2$

$= 2x (x - 2) - 1 (x - 2)$

$= (2x - 1) (x - 2)$

On substituting in equation 1, we get:

$p(x) = (x - 1) (2x - 1) (x - 2)$

Example 2: Factorize $x^3 - 23x^2 + 142x - 120$.

Solution:

Let $p(x) = x^3 - 23x^2 + 142x - 120$
Let us take $x = 1$ and find the value of $p(x)$.

$p(1)= 1^3 - 23 \times 1^2 + 142 \times 1 - 120$

$= 1 - 23 + 142 - 120$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x)= x^3 - 23x^2 + 142x - 120$

$= x^3 - x^2 - 22x^2 + 22x + 120x - 120$
$= x^2 (x -1) - 22x (x - 1) + 120 (x - 1)$

$= (x - 1) (x^2 - 22x + 120)$ ... (1)

Next, we factorize $x^2 - 22x + 120$ by splitting its middle term. The middle term is −22. We have to find two numbers such that their product is 120 and their sum is 22. These two numbers are 12 and 10.

Thus, we have:

$x^2 - 22x + 120= x^2 - 12x - 10x + 120$

$= x (x - 12) - 10 (x - 12)$

$= (x - 12) (x - 10)$

On substituting in equation (1), we get:

$x^3 - 23x^2 - 142x - 120 = (x - 1) (x - 12) (x - 10)$

Hard

Example 1: Factorize the cubic polynomial $p(x) = 6x^3 + 5x^2 - 12x + 4$.

Solution:

We have $p(x) = 6x^3 + 5x^2 - 12x + 4$

Let us take $x = -2$ and then find the value of $p(x)$.

$p(-2) = 6 \times (-2)^3 + 5 \times (-2)^2 - 12 \times (-2) + 4$

$= -48 + 20 + 24 + 4$

$= 0$

Thus, $x + 2$ is a factor of $p(x)$.

Now, let us divide $p(x)$ by $x + 2$.

$$
\begin{array}{r|l}
 & 6x^2 - 7x + 2 \\
x+2 & 6x^3 + 5x^2 - 12x + 4 \\
 & 6x^3 + 12x^2 \\
 & - \quad - \\
\hline
 & -7x^2 - 12x \\
 & -7x^2 - 14x \\
 & + \quad + \\
\hline
 & 2x + 4 \\
 & 2x + 4 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$\therefore 6x^3 + 5x^2 - 12x + 4 = (x + 2)(6x^2 - 7x + 2)$

$= (x + 2)(6x^2 - 4x - 3x + 2)$

$= (x + 2)[2x(3x - 2) - 1(3x - 2)]$

$= (x + 2)(3x - 2)(2x - 1)$

Using Identity for Square of Sum of Three Terms

Algebraic Identity:
$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

When we solve an **algebraic equation**, we get the values of the variables present in it. When an algebraic equation is valid for all values of its variables, it is called an **algebraic identity**.

So, an algebraic identity is a relation that holds true for all possible values of its variables. We can use algebraic identities to expand, factorise and evaluate various algebraic expressions.

Many algebraic identities are used in mathematics. One such identity is$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. In this lesson, we will study this identity and solve some examples based on it.

Proof of the Identity

Let us prove the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

We can write $(x + y + z)^2$ as :

$(a + z)^2$, where $a = x + y$

$= a^2 + 2az + z^2$ [Using the identity $(x + y)^2 = x^2 + 2xy + y^2$]

$= (x + y)^2 + 2(x + y) z + z^2$ (Substituting the value of a)

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$ (Using the identity $(x + y)^2 = x^2 + 2xy + y^2$)

$\therefore$ **$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$**

The above identity holds true for all values of the variables present in it. Let us verify this by substituting random values for the variables x, y and z.

If $x = 2$, $y = 3$ and $z = 4$, then:

$(2 + 3 + 4)^2 = 2^2 + 3^2 + 4^2 + 2 \times 2 \times 3 + 2 \times 3 \times 4 + 2 \times 4 \times 2$

$\Rightarrow 9^2 = 4 + 9 + 16 + 12 + 24 + 16$

$\Rightarrow 81 = 81$

$\Rightarrow$ LHS = RHS

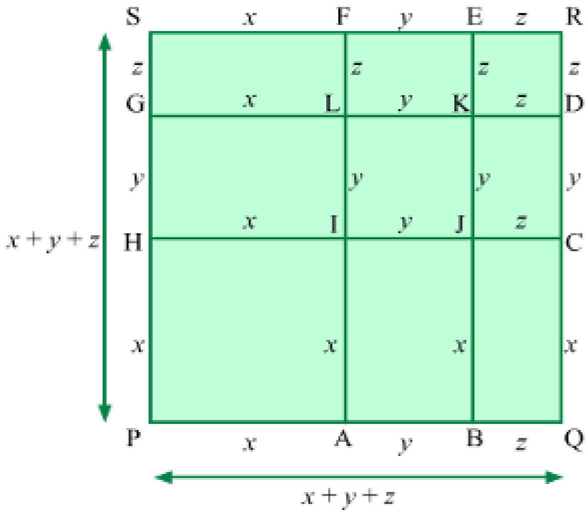
Thus, we see that the identity holds true for random values of the variables present in it.

Let us now use this identity to expand, factorise and evaluate various algebraic expressions.

Deriving Identity Geometrically

The identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ can also be derived with the help of geometrical construction.

The steps of construction are as follows:



(1) Draw a square PQRS of side measuring $(x + y + z)$ taking any convenient values of x, y and z.

(2) Mark two points A and B on side PQ such that $l(\text{PA}) = x$ and $l(\text{AB}) = y$. Thus, $l(\text{BQ}) = z$. Also, mark two points H and G on side PS such that $l(\text{PH}) = x$ and $l(\text{HG}) = y$. Thus, $l(\text{GS}) = z$.

(3) From points A and B, draw segments AF and BE parallel to side PS and intersecting RS at F and E respectively.

(4) From points H and G, draw segments HC and GD parallel to side PQ and intersecting QR at C and D respectively.

From the figure, it can be observed that

Area of square PQRS = Sum of areas of squares PAIH, IJKL and KDRE + Sum of areas of rectangles ABJI, BQCJ, JCDK, HILG, LKEF and GLFS

$\Rightarrow (x + y + z)^2 = (x^2 + y^2 + z^2) + (xy + zx + yz + xy + yz + zx)$

$\Rightarrow$**$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$**

Solved Examples

Easy

Example 1: Expand the following expressions.

i) $(ab - bc + ca)^2$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

Solution:

i) $(ab - bc + ca)^2$

On comparing the expression $(ab - bc + ca)^2$ with $(x + y + z)^2$, we get:

$x = ab, y = -bc$ and $z = ca$

On using the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get:

$(ab)^2 + (-bc)^2 + (ca)^2 + 2(ab)(-bc) + 2(-bc)(ca) + 2(ca)(ab)$

$= a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

$\therefore (ab - bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

On comparing the expression $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$ with $(a + b + c)^2$, we get:

$a = \frac{1}{2}x$, $b = -\frac{2}{3}y$ and $c = -\frac{3}{4}$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, we get:

$$\left(\frac{1}{2}x\right)^2+\left(-\frac{2}{3}y\right)^2+\left(-\frac{3}{4}\right)^2+2\left(\frac{1}{2}x\right)\left(-\frac{2}{3}y\right)+2\left(-\frac{2}{3}y\right)\left(-\frac{3}{4}\right)+2\left(-\frac{3}{4}\right)\left(\frac{1}{2}x\right)$$

$$\Rightarrow\frac{1}{4}x^2+\frac{4}{9}y^2+\frac{9}{16}-\frac{2}{3}xy+y-\frac{3}{4}x$$

$$\therefore\left(\frac{1}{2}x-\frac{2}{3}y-\frac{3}{4}\right)^2=\frac{1}{4}x^2+\frac{4}{9}y^2+\frac{9}{16}-\frac{2}{3}xy+y-\frac{3}{4}x$$

Example 2: Expand the expression $(xy + yz + zx)^2$.

Solution:

On comparing the expression $(xy + yz + zx)^2$ with $(a + b + c)^2$, we get:

$a = xy$, $b = yz$ and $c = zx$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,we get:

$(xy)^2 + (yz)^2 + (zx)^2 + 2\ (xy)\ (yz) + 2\ (yz)\ (zx) + 2\ (zx)\ (xy)$

$= x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

$\therefore\ (xy + yz + zx)^2 = x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

Medium

Example 1: Factorize the following expressions.

i)$8x^2 + 12y^2 - 8\sqrt{6}\,xy + 12\sqrt{6}\,x - 36y + 27$

ii)$8x^4 + 4\sqrt{2}\,x^3 + 25x^2 + 6\sqrt{2}\,x + 18$

Solution:

i)$8x^2 + 12y^2 - 8\sqrt{6}\,xy + 12\sqrt{6}\,x - 36y + 27$

$= 8x^2 + 12y^2 + 27 - 8\sqrt{6}\,xy - 36y + 12\sqrt{6}\,x$

$= (2\sqrt{2}\,x)^2 + (-2\sqrt{3}\,y)^2 + (3\sqrt{3})^2 + 2(2\sqrt{2}\,x)(-2\sqrt{3}\,y) + 2(-2\sqrt{3}\,y)(3\sqrt{3}) + 2(3\sqrt{3})(2\sqrt{2}\,x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,where $a = 2\sqrt{2}x$,
$b = -2\sqrt{3}y$ and $c = 3\sqrt{3}$, we are left with $(2\sqrt{2}x - 2\sqrt{3}y + 3\sqrt{3})^2$

$\therefore 8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27 = (2\sqrt{2}x - 2\sqrt{3}y + 3\sqrt{3})^2$

ii) $8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18$

$= 8x^4 + 25x^2 + 18 + 4\sqrt{2}x^3 + 6\sqrt{2}x$

$= 8x^4 + x^2 + 18 + 4\sqrt{2}x^3 + 6\sqrt{2}x + 24x^2$ ($\because 25x^2 = x^2 + 24x^2$)

$= (2\sqrt{2}x^2)^2 + (x)^2 + (3\sqrt{2})^2 + 2(2\sqrt{2}x^2)(x) + 2(x)(3\sqrt{2}) + 2(3\sqrt{2})(2\sqrt{2}x^2)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,where $a = 2\sqrt{2}x^2$,
$b = x$ and $c = 3\sqrt{2}$, we are left with $(2\sqrt{2}x^2 + x + 3\sqrt{2})^2$

$\therefore 8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18 = (2\sqrt{2}x^2 + x + 3\sqrt{2})^2$

Example 2: Find the value of the expression $4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$ for $x = 3$, $y = 4$ and
$z = 5$ without substituting the values of the variables in the expression.

Solution:

$4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$

$= (2x)^2 + (-3y)^2 + (4z)^2 + 2(2x)(-3y) + 2(-3y)(4z) + 2(4z)(2x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,where $a = 2x$,
$b = -3y$ and $c = 4z$, we are left with $(2x - 3y + 4z)^2$

It is given that $x = 3$, $y = 4$ and $z = 5$.

On substituting the values of x, y and z, we get:

$(2 \times 3 - 3 \times 4 + 4 \times 5)^2$

$= (6 - 12 + 20)^2$

$= 14^2$

$= 196$

$\therefore 4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx = 196$

Hard

Example 1: Find the value of $ab + bc + ca$, where $a + b + c = 1$ and $a^2 + b^2 + c^2 = 29$.

Solution:

We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow (1)^2 = 29 + 2(ab + bc + ca)$

$\Rightarrow 1 - 29 = 2(ab + bc + ca)$

$\Rightarrow -28 = 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca = -\frac{28}{2}$

$\Rightarrow \therefore ab + bc + ca = -14$

Example 2: If $(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x(6 + a) + 2y(2 - a) - 8z + 2(xy + yz + xz) - 8(3 - a)$

$= (x + y + z)^2$, then find the value of a.

Solution:

$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x(6 + a) + 2y(2 - a) - 8z + 2(xy + yz + xz) - 8(3 - a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 8z + 2xy + 2yz + 2xz - 24 + 8a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 12z + 4z + 2xy + 2yz + 2xz - 24 + 12a - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2xy - 12x + 4y - 24 + 2yz - 2ay - 12z + 12a + 2xz - 2ax + 4z - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2(xy - 6x + 2y - 12) + 2(yz - ay - 6z + 6a) + 2(xz - ax + 2z - 2a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2[x(y - 6) + 2(y - 6)] + 2[y(z - a) - 6(z - a)] + 2[x(z - a) + 2(z - a)]$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2\,(x + 2)\,(y - 6) + 2\,(y - 6)\,(z - a) + 2\,(x + 2)\,(z - a)$

$= [(x + 2) + (y - 6) + (z - a)]^2$

$= (x + y + z - 4 - a)^2$

It is given that
$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x\,(6 + a) + 2y\,(2 - a) - 8z + 2\,(xy + yz + xz) - 8\,(3 - a)$

$= (x + y + z)^2$

$\Rightarrow (x + y + z - 4 - a)^2 = (x + y + z)^2$

$\Rightarrow -4 - a = 0$

$\Rightarrow \therefore a = -4$

Using Identities for Cube of Sum or Difference of Two Terms

Algebraic Identities:
$\mathbf{(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)}$ **and** $\mathbf{(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)}$

Consider the number '999'. Suppose we have to calculate its cube. One way to find the cube is to multiply 999 by itself three times. However, this method is tedious and, therefore, prone to error.

Here is another way to solve the problem. Let us write 999^3 as $(1000 - 1)^3$. We have thus changed the number into the form $(x - y)^3$. Now, the expansion of $(x - y)^3$ will give the cube of 999. The required calculation will be easy since the values of x and y are simple numbers whose multiplication is also simple.

Thus, we see algebraic identities help make calculations simpler and less tedious. In this lesson, we will study the identities $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$. We will also solve some examples based on them.

Understanding the Identities

We have the two algebraic identities as follows:

- $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ OR $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$
- $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$ OR $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$

The above identities hold true for all values of the variables present in them. Let us verify this by substituting random values for the variables x and y in the first identity.

If $x = 2$ and $y = 3$, then:

$(2 + 3)^3 = 2^3 + 3^3 + 3 \times 2 \times 3 \times (2 + 3)$

$\Rightarrow 5^3 = 8 + 27 + 18 \times 5$

$\Rightarrow 125 = 8 + 27 + 90$

$\Rightarrow 125 = 125$

$\Rightarrow$ LHS = RHS

Thus, we see that the first identity holds true for random values of the variables present in it. We can prove the same for the second identity as well.

Here are some other ways in which the two identities can be represented

- $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$
- $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$ OR $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

Proof of the Identities:
$(x + y)^3 = \ldots$ and $x^3 + y^3 = \ldots$

Let us prove the identity $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$ OR $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

We can write $(x + y)^3$ as:

$(x + y)\,(x + y)^2$

$= (x + y)\,(x^2 + 2xy + y^2)$

$= x^3 + 2x^2y + 2xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + y^3 + 3x^2y + 3xy^2$

$\therefore$ **$(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$**

$\Rightarrow$ **$(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$** … (1)

Let us prove the identity $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$

We can rewrite equation 1 as:

$\mathbf{x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)}$

$\Rightarrow x^3 + y^3 = (x + y)\,[(x + y)^2 - 3xy]$

$\Rightarrow x^3 + y^3 = (x + y)\,(x^2 + 2xy + y^2 - 3xy)$

$\Rightarrow \mathbf{x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)}$

Proof of the Identities:
$(x - y)^3 = \ldots$ and $x^3 - y^3 = \ldots$

Let us prove the identity $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$ OR $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

We can write $(x - y)^3$ as:

$(x - y)\,(x - y)^2$

$= (x - y)\,(x^2 - 2xy + y^2)$

$= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3$

$= x^3 - y^3 - 3x^2y + 3xy^2$

$\therefore \mathbf{(x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2}$

$\Rightarrow \mathbf{(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)}$ … (1)

Let us prove the identity $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$ OR $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

We can rewrite equation 1 as:

$\mathbf{x^3 - y^3 = (x-y)^3 + 3xy\,(x-y)}$

$\Rightarrow x^3 - y^3 = (x - y)\,[(x - y)^2 + 3xy]$

$\Rightarrow x^3 - y^3 = (x - y)\,(x^2 - 2xy + y^2 + 3xy)$

$\Rightarrow \mathbf{x^3 - y^3 = (x-y)\,(x^2 + xy + y^2)}$

Example Based on the Identity $x^3 - y^3 = \ldots$

Solved Examples

Easy

Example 1: Factorise the following expressions.

i) $a^3 - 125b^3 - 15a^2b + 75ab^2$

ii) $27p^3 + 125q^3$

Solution:

i) $a^3 - 125b^3 - 15a^2b + 75ab^2$

$= (a)^3 - (5b)^3 - 15ab\,(a - 5b)$

$= (a)^3 - (5b)^3 - 3 \times a \times 5b\,(a - 5b)$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$, where $x = a$ and $y = 5b$, we are left with $(a - 5b)^3$

$\therefore a^3 - 125b^3 - 15a^2b + 75ab^2 = (a - 5b)^3$

ii) $27p^3 + 125q^3$

$= (3p)^3 + (5q)^3$

On using the identity $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$, where $x = 3p$ and $y = 5q$, we get:

$(3p + 5q)\,[(3p)^2 - (3p)\,(5q) + (5q)^2]$

$= (3p + 5q)\,(9p^2 - 15pq + 25q^2)$

$\therefore 27p^3 + 125q^3 = (3p + 5q)\,(9p^2 - 15pq + 25q^2)$

Example 2: Evaluate the following expressions using identities.

i) 1003^3

ii) 98^3

Solution:

i) We can write 1003^3 as:

$(1000 + 3)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$, where $x = 1000$ and $y = 3$, we get:

$(1000 + 3)^3 = 1000^3 + 3^3 + 3 \times 1000 \times 3 \times (1000 + 3)$

$= 1000000000 + 27 + 9000 \times (1000 + 3)$

$= 1000000000 + 27 + 9000000 + 27000$

$= 1009027027$

ii)We can write 98^3 as:

$(100 - 2)^3$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$, where $x = 100$ and $y = 2$, we get:

$(100 - 2)^3 = 100^3 - 2^3 - 3 \times 100 \times 2 \times (100 - 2)$

$= 1000000 - 8 - 600 \times (100 - 2)$

$= 1000000 - 8 - 60000 + 1200$

$= 941192$

Medium

Example 1: Expand the following expressions.

i) $\left(\frac{x}{a} + \frac{y}{b}\right)^3$

ii) $(2x + 5y)^3 - (2x - 5y)^3$

Solution:

i) $\left(\frac{x}{a} + \frac{y}{b}\right)^3$

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$, where $p = \frac{x}{a}$ and $q = \frac{y}{b}$, we get:

$$\left(\frac{x}{a}+\frac{y}{b}\right)^3=\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+3\times\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)$$
$$=\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{3xy}{ab}\left(\frac{x}{a}+\frac{y}{b}\right)$$
$$=\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{3x^2y}{a^2b}+\frac{3xy^2}{ab^2}$$

ii) $(2x + 5y)^3 - (2x - 5y)^3$

We have two terms in the given expression—$(2x + 5y)^3$ and $(2x - 5y)^3$.

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq\,(p + q)$, where $p = 2x$ and $q = 5y$, we get:

$(2x + 5y)^3 = (2x)^3 + (5y)^3 + 3 \times (2x)\,(5y)\,(2x + 5y)$

$= 8x^3 + 125y^3 + 30xy\,(2x + 5y)$

$= 8x^3 + 125y^3 + 60x^2y + 150xy^2$

Similarly, on using the identity $(p - q)^3 = p^3 - q^3 - 3pq\,(p - q)$, we get

$(2x - 5y)^3 = (2x)^3 - (5y)^3 - 3 \times (2x)\,(5y)\,(2x - 5y)$

$= 8x^3 - 125y^3 - 30xy\,(2x - 5y)$

$= 8x^3 - 125y^3 - 60x^2y + 150xy^2$

So,

$(2x + 5y)^3 - (2x - 5y)^3 = 8x^3 + 125y^3 + 60x^2y + 150xy^2 - [8x^3 - 125y^3 - 60x^2y + 150xy^2]$

$= 8x^3 + 125y^3 + 60x^2y + 150xy^2 - 8x^3 + 125y^3 + 60x^2y - 150xy^2$

$= 250y^3 + 120x^2y$

Alternate method

On using the identity $p^3 - q^3 = (p - q)\,(p^2 + pq + q^2)$, where $p = (2x + 5y)$ and $q = (2x - 5y)$, we get:

$(2x + 5y)^3 - (2x - 5y)^3$

$= [(2x + 5y) - (2x - 5y)]\,[(2x + 5y)^2 + (2x + 5y)\,(2x - 5y) + (2x - 5y)^2]$

$= (2x + 5y - 2x + 5y)\,(4x^2 + 20xy + 25y^2 + 4x^2 + 10xy - 10xy - 25y^2 + 4x^2 - 20xy + 25y^2)$

$= 10y\,(12x^2 + 25y^2)$

$= 120x^2y + 250y^3$

Example 2:The side of a cube is a. If each side of the cube is increased by $\frac{b}{5}$, then by how much does its volume increase?

Solution:

Let the side of the cube be a.

Original volume of the cube = $a \times a \times a = a^3$

After the increase, each side becomes $\left(a+\frac{b}{5}\right)$.

New volume = $\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)=\left(a+\frac{b}{5}\right)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$, where $x = a$ and $y = \frac{b}{5}$, we get:

$$\begin{aligned}\left(a+\frac{b}{5}\right)^3 &= a^3+\left(\frac{b}{5}\right)^3+3(a)\left(\frac{b}{5}\right)\left(a+\frac{b}{5}\right)\\ &= a^3+\frac{b^3}{125}+\frac{3ab}{5}\times\left(a+\frac{b}{5}\right)\\ &= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}\end{aligned}$$

Now,

Increase in volume = New volume − Original volume

$$= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}-a^3$$
$$= \frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$$

Thus, the volume of the cube increases by $\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$.

Hard

Example 1: Find the values of the following expressions.

i) $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=5$

ii) $8y^3-\frac{27}{y^3}$ when $4y^2+\frac{9}{y^2}=37$

iii) $125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

Solution:

i) $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=5$

We have $x+\frac{1}{x}=5$

On cubing both sides, we get:

$$\left(x+\frac{1}{x}\right)^3=5^3$$

On using the identity $(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$, where $a = x$ and $b = \frac{1}{x}$, we get:

$$\left(x+\frac{1}{x}\right)^3=125$$

$$\Rightarrow x^3+\frac{1}{x^3}+3(x)\left(\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=125$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-3\left(x+\frac{1}{x}\right)$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-3\times 5$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-15$$

$$\Rightarrow x^3+\frac{1}{x^3}=110$$

ii) $8y^3-\frac{27}{y^3}$ when $4y^2+\frac{9}{y^2}=37$

We have $4y^2+\frac{9}{y^2}=37$

$$\Rightarrow (2y)^2+\left(\frac{3}{y}\right)^2=37$$

On using the identity $a^2 + b^2 = (a - b)^2 + 2ab$, where $a = 2y$ and $b = \frac{3}{y}$, we get:

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2+2(2y)\left(\frac{3}{y}\right)=37$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2+12=37$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2=37-12$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2=25$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)=5 \qquad ...(1)$$

On cubing both sides, we get:

$$\left(2y-\frac{3}{y}\right)^3=5^3$$

On using the identity $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we get:

$$(2y)^3-\left(\frac{3}{y}\right)^3-3(2y)\left(\frac{3}{y}\right)\left(2y-\frac{3}{y}\right)=125$$
$$\Rightarrow 8y^3-\frac{27}{y^3}-18\left(2y-\frac{3}{y}\right)=125$$

On using equation 1, we get:

$$8y^3-\frac{27}{y^3}-18\times 5=125$$
$$\Rightarrow 8y^3-\frac{27}{y^3}=125+90$$
$$\Rightarrow 8y^3-\frac{27}{y^3}=215$$

iii) $125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

We have $5x - 3y = 1$

On cubing both sides, we get:

$(5x - 3y)^3 = 1^3$

On using the identity $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, where $a = 5x$ and $b = 3y$, we get:

$(5x)^3 - (3y)^3 - 3(5x)(3y)(5x - 3y) = 1$

$\Rightarrow 125x^3 - 27y^3 - 45xy(5x - 3y) = 1$

On substituting the values of xy and $(5x - 3y)$,we get:

$125x^3 - 27y^3 - 45 \times 6 \times 1 = 1$

$\Rightarrow 125x^3 - 27y^3 = 1 + 270$

$\Rightarrow 125x^3 - 27y^3 = 271$

Example 2: If $x + y = 8$ and $x^2 + y^2 = 42$, then find the value of $x^3 + y^3$.

Solution:

It is given that $x + y = 8$

$\Rightarrow (x + y)^2 = 8^2$

$\Rightarrow x^2 + y^2 + 2xy = 64$

$\Rightarrow 42 + 2xy = 64$ $(\because x^2 + y^2 = 42)$

$\Rightarrow 2xy = 64 - 42$

$\Rightarrow 2xy = 22$

$\Rightarrow xy = \frac{22}{2}$

$\Rightarrow \therefore xy = 11$

Now, $(x + y)^3 = 8^3$

$\Rightarrow x^3 + y^3 + 3xy\,(x + y) = 512$

$\Rightarrow x^3 + y^3 + 3 \times 11 \times 8 = 512$

$\Rightarrow x^3 + y^3 + 264 = 512$

$\Rightarrow x^3 + y^3 = 512 - 264$

$\Rightarrow \therefore x^3 + y^3 = 248$

Example 3: Prove that $\frac{0.77 \times 0.77 \times 0.77 + 0.23 \times 0.23 \times 0.23}{0.77 \times 0.77 - 0.77 \times 0.23 + 0.23 \times 0.23} = 1$.

Solution:

$$\frac{0.77\times0.77\times0.77+0.23\times0.23\times0.23}{0.77\times0.77-0.77\times0.23+0.23\times0.23}$$

$$=\frac{(0.77)^3+(0.23)^3}{(0.77)^2-0.77\times0.23+(0.23)^2}$$

$$=\frac{x^3+y^3}{x^2-xy+y^2}, \text{ where } x=0.77 \text{ and } y=0.23$$

$$=\frac{(x+y)(x^2-xy+y^2)}{(x^2-xy+y^2)}$$

$$=(0.77+0.23)$$

$$=1$$

Solving Problems Using the Identity (x + y + z) ($x^2 + y^2 + z^2 - xy - yz - zx$)

Algebraic Identity:
$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Algebraic identities help us solve problems with ease and in minimum time. Say, for example, we need to find the value of $(-32^3 + 15^3 + 17^3)$. One may solve this problem by calculating the cube of each of the given numbers and then adding and subtracting the values so obtained. This method is easy in cases where we are dealing with small numbers. However, when big numbers are involved (as in the present case), this method proves to be tedious.

A simpler and less time-consuming way of solving the above problem is to use an appropriate algebraic identity. In the given expression, we find that −32 + 15 + 17 = 0. So, we need to state an identity under the condition $x + y + z = 0$.

In this lesson, we will focus on the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)$ $(x^2 + y^2 + z^2 - xy - yz - zx)$ and its expansion under the condition $x + y + z = 0$. We will also solve examples based on the same.

Understanding the Identity

We have the algebraic identity as follows:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Or

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$$

The above identity holds true for all values of the variables present in it. Let us verify this by substituting random values for the variables x, y and z.

If $x = 1$, $y = 2$ and $z = 3$, then:

$1^3 + 2^3 + 3^3 - 3 \times 1 \times 2 \times 3 = (1 + 2 + 3)\ (1^2 + 2^2 + 3^2 - 1 \times 2 - 2 \times 3 - 3 \times 1)$

$\Rightarrow 1 + 8 + 27 - 18 = 6\ (1 + 4 + 9 - 2 - 6 - 3)$

$\Rightarrow 18 = 6 \times 3$

$\Rightarrow 18 = 18$

$\Rightarrow$ LHS = RHS

Thus, we see that the identity holds true for random values of the variables present in it.

Proof of the Identity

Let us prove the identity

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)$

Or

$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}\ (x + y + z)\ [(x - y)^2 + (y - z)^2 + (z - x)^2]$

We can write $x^3 + y^3 + z^3 - 3xyz$ as:

$(x^3 + y^3) + z^3 - 3xyz$

$= [(x + y)^3 - 3xy(x + y)] + z^3 - 3xyz$

$= a^3 - 3axy + z^3 - 3xyz$, where $a = x + y$

$= (a^3 + z^3) - 3axy - 3xyz$

$= (a + z)\ (a^2 - az + z^2) - 3xy(a + z)$

$= (a + z)\ (a^2 - az + z^2 - 3xy)$

$= (x + y + z)\ [(x + y)^2 - (x + y)z + z^2 - 3xy]$

$= (x + y + z)\ (x^2 + y^2 + 2xy - zx - yz + z^2 - 3xy)$

$= (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)$

$\therefore$ $\mathbf{x^3 + y^3 + z^3 - 3xyz = (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)}$

On multiplying and dividing the above expanded form by 2, we get:

$\frac{1}{2} \times 2\ (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)$

$= \frac{1}{2}\ (x + y + z)\ (2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$

$= \frac{1}{2}\ (x + y + z)\ (x^2 + x^2 + y^2 + y^2 + z^2 + z^2 - 2xy - 2yz - 2zx)$

$= \frac{1}{2}\ (x + y + z)\ (x^2 + y^2 - 2xy + y^2 + z^2 - 2yz + z^2 + x^2 - 2zx)$

$= \frac{1}{2}\ (x + y + z)\ [(x - y)^2 + (y - z)^2 + (z - x)^2]$

$\therefore$ $\mathbf{x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}\ (x + y + z)\ [(x - y)^2 + (y - z)^2 + (z - x)^2]}$

Solved Examples

Easy

Example 1: Factorize the following expressions.

i) $125x^3 + 8y^3 + 27z^3 - 90xyz$

ii) $343p^3 - 64y^3 + 8 + 168py$

Solution:

i) $125x^3 + 8y^3 + 27z^3 - 90xyz$

$= (5x)^3 + (2y)^3 + (3z)^3 - 3\ (5x)\ (2y)\ (3z)$

On using the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)$, we get:

$(5x + 2y + 3z)\ [(5x)^2 + (2y)^2 + (3z)^2 - (5x)\ (2y) - (2y)\ (3z) - (3z)\ (5x)]$

$= (5x + 2y + 3z)\ (25x^2 + 4y^2 + 9z^2 - 10xy - 6yz - 15xz)$

$\therefore 125x^3 + 8y^3 + 27z^3 - 90xyz = (5x + 2y + 3z)\ (25x^2 + 4y^2 + 9z^2 - 10xy - 6yz - 15xz)$

ii) $343p^3 - 64y^3 + 8 + 168py$

$= (7p)^3 + (-4y)^3 + (2)^3 - 3\ (7p)\ (-4y)\ (2)$

On using the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)$, we get:

$[7p + (-4y) + 2]\ [(7p)^2 + (-4y)^2 + (2)^2 - (7p)\ (-4y) - (-4y)\ (2) - (2)\ (7p)]$

$= (7p - 4y + 2)\ (49p^2 + 16y^2 + 4 + 28py + 8y - 14p)$

$\therefore 343p^3 - 64y^3 + 8 + 168py = (7p - 4y + 2)\ (49p^2 + 16y^2 + 4 + 28py + 8y - 14p)$

Medium

Example 1: If $a + b + c = 10$ and $ab + bc + ca = 31$, then find the value of $a^3 + b^3 + c^3 - 3abc$.

Solution: We know that $a^3 + b^3 + c^3 - 3abc = (a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ca)$

Or, $a^3 + b^3 + c^3 - 3abc = (a + b + c)\ [a^2 + b^2 + c^2 - (ab + bc + ca)]$ ... (1)

It is given that $a + b + c = 10$ and $ab + bc + ca = 31$.

Let us find the value of $a^2 + b^2 + c^2$.

We have the identity:

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow (10)^2 = a^2 + b^2 + c^2 + 2 \times 31$

$\Rightarrow 100 = a^2 + b^2 + c^2 + 62$

$\Rightarrow a^2 + b^2 + c^2 = 100 - 62$

$\Rightarrow a^2 + b^2 + c^2 = 38$

On substituting all the values in equation 1, we obtain:

$a^3 + b^3 + c^3 - 3abc = 10 \times (38 - 31)$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 10 \times 7$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 70$

Example 2: Factorise the expression $\frac{x^3}{8} - \frac{27}{x^3} - \frac{11}{2}$.

Solution:

We have

$$\frac{x^3}{8} - \frac{27}{x^3} - \frac{11}{2}$$

$$= \frac{x^3}{8} - \frac{27}{x^3} - 1 - \frac{9}{2}$$

$$= \left(\frac{x}{2}\right)^3 + \left(-\frac{3}{x}\right)^3 + (-1)^3 - 3\left(\frac{x}{2}\right)\left(-\frac{3}{x}\right)(-1)$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\left(\frac{x}{2}\right)^2 + \left(-\frac{3}{x}\right)^2 + (-1)^2 - \left(\frac{x}{2}\right)\left(-\frac{3}{x}\right) - \left(-\frac{3}{x}\right)(-1) - (-1)\left(\frac{x}{2}\right)\right\}$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\frac{x^2}{4} + \frac{9}{x^2} + 1 + \frac{3}{2} - \frac{3}{x} + \frac{x}{2}\right\}$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\frac{x^2}{4} + \frac{x}{2} + \frac{5}{2} - \frac{3}{x} + \frac{9}{x^2}\right\}$$

Case I of the Identity

A special case for the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ is given below.

Case: When $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$.

Proof: We have,

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On substituting $x + y + z = 0$, we obtain

$x^3 + y^3 + z^3 - 3xyz = 0 \times (x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^3 + y^3 + z^3 - 3xyz = 0$

$\Rightarrow x^3 + y^3 + z^3 = 3xyz$

Using this condition, we can factorize and find the values of many complex expressions.

Solved Examples

Easy

Example 1: Without actually calculating the cubes, find the value of each of the following expressions.

i) $(0.2)^3 - (0.5)^3 + (0.3)^3$

ii) $-12^3 + 25^3 - 13^3$

Solution:

i) We can write $(0.2)^3 - (0.5)^3 + (0.3)^3$ as $(0.2)^3 + (-0.5)^3 + (0.3)^3$.

Let us consider $x = 0.2$, $y = -0.5$ and $z = 0.3$.

Now, $x + y + z = 0.2 - 0.5 + 0.3 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

On substituting the values of x, y and z, we obtain:

$(0.2)^3 + (-0.5)^3 + (0.3)^3 = 3 \times 0.2 \times (-0.5) \times 0.3$

$\Rightarrow (0.2)^3 - (0.5)^3 + (0.3)^3 = -0.09$

ii) We can write $-12^3 + 25^3 - 13^3$ as $(-12)^3 + 25^3 + (-13)^3$.

Let us consider $x = -12$, $y = 25$ and $z = -13$.

Now, $x + y + z = -12 + 25 - 13 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

On substituting the values of x, y and z, we obtain:

$(-12)^3 + 25^3 + (-13)^3 = 3 \times (-12) \times 25 \times (-13)$

$\Rightarrow -12^3 + 25^3 - 13^3 = 11700$

Medium

Example 1: Find the value of the expression $8x^3 + 27y^3 - 64z^3$ without directly substituting the values
$x = 3, y = 2$ and $z = 3$.

Solution: We can write $8x^3 + 27y^3 - 64z^3$ as $(2x)^3 + (3y)^3 + (-4z)^3$.

For the given values of x, y and z, we get:

$(2x) + (3y) + (-4z) = 2 \times 3 + 3 \times 2 - 4 \times 3$

$\Rightarrow (2x) + (3y) + (-4z) = 6 + 6 - 12$

$\Rightarrow (2x) + (3y) + (-4z) = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we have:

$(2x)^3 + (3y)^3 + (-4z)^3 = 3\,(2x)\,(3y)\,(-4z)$

$\Rightarrow 8x^3 + 27y^3 - 64z^3 = -72xyz$

On substituting the values of x, y and z, we obtain:

$8x^3 + 27y^3 - 64z^3 = -72 \times 3 \times 2 \times 3$

$\Rightarrow 8x^3 + 27y^3 - 64z^3 = -1296$

Example 1: Show that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$.

Solution: We will factorize the numerator and the denominator separately.

We have the numerator as $(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3$

Let us consider $x = a^2 - b^2, y = b^2 - c^2$ and $z = c^2 - a^2$.

Now, $x + y + z = a^2 - b^2 + b^2 - c^2 + c^2 - a^2 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we obtain:

$(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3\,(a^2 - b^2)\,(b^2 - c^2)\,(c^2 - a^2)$

On using the identity $x^2 - y^2 = (x + y)\,(x - y)$, we get:

$(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3\,(a - b)\,(a + b)\,(b - c)\,(b + c)\,(c - a)\,(c + a)$

We have the denominator as $(a - b)^3 + (b - c)^3 + (c - a)^3$.

Let us consider $x = a - b$, $y = b - c$ and $z = c - a$.

Now, $x + y + z = a - b + b - c + c - a = 0$

Again, if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we obtain:

$(a - b)^3 + (b - c)^3 + (c - a)^3 = 3\,(a - b)\,(b - c)\,(c - a)$

On putting back the numerator and the denominator, we get:

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=\frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$$

$$\Rightarrow \frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$$

Case II of the Identity

One more special case of the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$ is there which is explained below.

Case: When $x + y + z \neq 0$ and $x^3 + y^3 + z^3 - 3xyz = 0$ then $x = y = z$.

Proof: We have,

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

On substituting $x^3 + y^3 + z^3 - 3xyz = 0$, we obtain

$0 = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^2 + y^2 + z^2 - xy - yz - zx = 0 \qquad (x + y + z \neq 0)$

$\Rightarrow \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

$\Rightarrow [(x - y)^2 + (y - z)^2 + (z - x)^2] = 0 \qquad (x + y + z \neq 0)$

Since, the sum of non negative terms such as $(x - y)^2$, $(y - z)^2$ and $(z - x)^2$ is 0, each term is 0.

$\therefore\ (x - y)^2 = 0$, $(y - z)^2 = 0$ and $(z - x)^2 = 0$

$\Rightarrow\ x - y = 0, y - z = 0$ and $z - x = 0$

$\Rightarrow\ x = y, y = z$ and $z = x$

$\Rightarrow\ x = y = z$

This condition can be very helpful to factorize and find the values of many complex expressions.

Example Based on the Case II of the Identity

Example : If $2a = 3b = 4c = 24$ then without actually calculating the cubes of a, b and c, find the value of $8a^3 + 27b^3 + 64c^3$.

Solution:We have,

$2a = 3b = 4c = 24$

$\Rightarrow a = 12, b = 8$ and $c = 6$

Also, $2a + 3b + 4c = 72 \neq 0$

Therefore,

$(2a)^3 + (3b)^3 + (4c)^3 - 3(2a)(3b)(4c) = 0$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 72abc$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 72 \times 12 \times 8 \times 6$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 41472$